

UNIFIED FIELD THEORY AND PRINCIPLE OF REPRESENTATION INVARIANCE

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ABSTRACT. This is part of a research program to establish a unified field model for interactions in nature. One main objective of this article is to postulate a new principle of representation invariance (PRI), and to refine the unified field model, derived using the principle of interaction dynamics (PID). Intuitively, PID takes the variation of the action functional under energy-momentum conservation constraint, and PRI requires that all $SU(N)$ gauge theories should be invariant under transformations of different representations of $SU(N)$. With PRI, we are able to substantially reduce the number of to-be-determined parameters in the unified field model to two $SU(2)$ and $SU(3)$ constant vectors $\{\alpha_\mu^w\}$ and $\{\alpha_k^s\}$, containing 11 parameters, which represent the portions distributed to the gauge potentials by the weak and strong charges g_w and g_s . This unified field model can be naturally decoupled to study individual interactions. The second objective is to explore the duality of strong interaction based on the new field equations, derived by applying PID and PRI to a standard QCD $SU(3)$ gauge action functional. The new field equations establish a natural duality between strong gauge fields $\{S_\mu^k\}$, representing the eight gluons, and eight bosonic scalar fields. One prediction of this duality is the existence of a Higgs type bosonic spin-0 particle with mass $m \geq 100\text{GeV}/c^2$. With the duality, we derive three levels of strong interaction potentials: the quark potential S_q , the nucleon/hadron potential S_n and the atom/molecule potential S_a . These potentials clearly demonstrates many features of strong interaction consistent with observations. In particular, these potentials offer a clear mechanisms for quark confinement, for asymptotic freedom, and for the van der Waals force. Also, in the nuclear level, the new potential is an improvement of the Yukawa potential. As the distance between two nucleons is increasing, the nuclear force corresponding to the nucleon potential S_n behaves as repelling, then attracting, then repelling again and diminishes, consistent with experimental observations. The third objective is to study the duality of weak interaction, and to derive the long overdue weak potential and force formula. The fourth objective is to try to address such questions as why our universe is as it is from the smallest elementary particles to stars, galaxies and the largest cosmos. We attempt to offer our view on this fundamental problem by introducing energy levels for leptons and quarks as well as for hadrons, and by exploring both essential characteristics of four forces derived using the unified field theory.

Key words and phrases. unified field equations, Principle of Interaction Dynamics, Principle of Representation Invariance, duality theory of strong interaction, duality theory of weak interaction, quark confinement, asymptotic freedom, Higgs mechanism, Higgs bosons, quark potential, nucleon potential, atom potential, weak interaction potential, strong interaction force formulas, weak interaction force formula, electroweak theory, van der Waals force, energy levels of leptons and quarks, energy levels of hadrons, force formulas for four interactions, stability of matter .

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1. INTRODUCTION

There are four forces/interactions in nature: the electromagnetic force, the strong force, the weak force and the gravitational force. Classical theories describing these interactions include the Einstein general theory of relativity, the quantum electromagnetic dynamics (QED) for electromagnetism, the Weinberg-Salam electroweak theory unifying weak and electromagnetic interactions [3, 14, 13], the quantum chromodynamics (QCD) for strong interaction, and the standard model, a $U(1) \otimes SU(2) \otimes SU(3)$ gauge theory, unifying all known interactions except gravity; see among many others [6].

This article is part of a research program initiated recently by the authors to derive a unified field theory coupling natural interactions [10, 12]. There are several main objectives of this article. The first objective is to postulate a new principle of representation invariance (PRI), and to refine the unified field model, derived using the principle of interaction dynamics (PID) [12]. The unified field equations, on the one hand, are used to study the coupling mechanism of interactions in nature, and on the other hand can be decoupled to study individual interactions, leading to both experimentally verified results and new predictions. The second objective is to establish a duality theory for strong interaction, and to derive three levels of strong interaction potentials: the quark potential S_q , the nucleon/hadron potential S_n and the atom/molecule potential S_a . These potentials clearly demonstrates many features of strong interaction consistent with observations, and offer, in particular, a clear mechanism for both quark confinement and asymptotic freedom. The third objective is to study the duality of weak interaction, and to derive such weak potential and force formula. The fourth objective is to offer our view on the structure and stability of matter, and to introduce the concept of energy levels for leptons and quarks, and for hadrons.

Hereafter we address the main motivations and ingredients of the study.

1. The original motivation is an attempt to developing gravitational field equations to provide a unified theory for dark energy and dark matter [10]. The key point is that due to the presence of dark energy and dark matter, the energy-momentum tensor of visible matter, T_{ij} , is no longer conserved. Namely,

$$\nabla^i T_{ij} \neq 0,$$

where ∇^i is the contra-variant derivative. Since the Euler-Lagrangian of the scalar curvature part of the Einstein-Hilbert functional is conserved (Bianchi identity), it can only be balanced by the conserved part of T_{ij} . Thanks to an orthogonal decomposition of tensor fields into conserved and gradient parts [10], the new gravitational field equations are given then by

$$(1.1) \quad R_{ij} - \frac{1}{2}g_{ij}R = -\frac{8\pi G}{c^4}T_{ij} - \nabla_i \nabla_j \varphi,$$

where $\varphi : M \rightarrow \mathbb{R}$ is a scalar function defined on the space-time manifold, whose energy density $\Phi = g^{ij}\nabla_i\nabla_j\varphi$ is conserved with mean zero:

$$(1.2) \quad \int_M \Phi \sqrt{-g} dx = 0.$$

Equivalently, (1.1) is the Euler-Lagrangian of the Einstein-Hilbert functional L_{EH} with energy-momentum conservation constraints:

$$(1.3) \quad (\delta L_{EH}, X) = 0 \quad \text{for } X = \{X_{ij}\} \text{ with } \nabla^i X_{ij} = 0.$$

As we have discussed in [10], the above gravitational field equations offer a unified theory for dark energy and dark matter, agreeable with all the general features/observations for both dark matter and dark energy.

2. The constraint Lagrangian action (1.3) leads us to postulate a general principle, which we call principle of interaction dynamics (PID), for deriving unified field equations coupling interactions in nature. Namely, for physical interactions with the Lagrangian action $L(g, A, \psi)$, the field equations are the Euler-Lagrangian of $L(g, A, \psi)$ with div_A -free constraint:

$$(1.4) \quad (\delta F(u_0), X) = \int_M \delta F(u_0) \cdot X \sqrt{-g} dx = 0 \quad \text{for } X \text{ with } \text{div}_A X = 0.$$

Here A is a set of vector fields representing gauge and mass potentials, ψ are the wave functions of particles, and div_A is defined by (2.1). It is clear that div_A -free constraint is equivalent to energy-momentum conservation.

3. We then derive in [12] the unified field equations coupling four interactions based on 1) the Einstein principle of general relativity (or Lorentz invariance) and the principle of equivalence, 2) the principle of gauge invariance, and 3) the PID. Naturally, the Lagrangian action functional is the combination of the Einstein-Hilbert action for gravity, the action of the $U(1)$ gauge field for electromagnetism, the standard $SU(2)$ Yang-Mills gauge action for the weak interactions, and the standard $SU(3)$ gauge action for the strong interactions. The unified model gives rise to a new mechanism for spontaneous gauge-symmetry breaking and for energy and mass generations with similar outcomes as the classical Higgs mechanism. One important outcome of the unified field equations is a natural duality between the interacting fields (g, A, W^a, S^k) , corresponding to graviton, photon, intermediate vector bosons W^\pm and Z and gluons, and the adjoint fields $(\Phi_\mu, \phi^0, \phi_w^a, \phi_s^k)$, which are all bosonic fields. The interaction of the bosonic particle field Φ and graviton leads to a unified theory of dark matter and dark energy and explains the acceleration of expanding universe.

4. It is classical that the electromagnetism is described by a $U(1)$ gauge field, the weak interactions are described by three $SU(2)$ gauge fields, and the strong interactions are described by eight $SU(3)$ gauge fields. In the same spirit as the Einstein principle of general relativity, physical laws should be independent of different representations of these Lie groups. Hence it is natural for us to postulate a general principle, which we call the principle of representation invariance (PRI):

Principle of Representation Invariance (PRI). *All $SU(N)$ gauge theories are invariant under general linear group $GL(\mathbb{C}^{N^2-1})$ transformations for generators of different representations of $SU(N)$. Namely, the actions of the gauge fields*

are invariant and the corresponding gauge field equations are covariant under the transformations.

5. The mathematical foundation of PRI is achieved by deriving a few mathematical results for representations of the Lie group $SU(N)$. In particular, for the Lie group $SU(N)$, generators of different representations transform under general linear group $GL(\mathbb{C}^{N^2-1})$. We show that the structural constants λ_{ab}^c of the generators of different representations should transfer as $(1, 2)$ -tensors. Consequently, we can construct an important $(0, 2)$ $SU(N)$ -tensor:

$$(1.5) \quad G_{ab} = \frac{1}{4N} \lambda_{ad}^c \lambda_{cb}^d,$$

which can be regarded as a Riemannian metric on the Lie group $SU(N)$.

Then for a set of $SU(N)$ ($N \geq 2$) gauge fields with $N^2 - 1$ vector fields A_μ^a and N spinor fields ψ^j , the following action functional is a unique functional which obeys the Lorentz invariance, the gauge invariance of the transformation (3.13), and is invariant under $GL(\mathbb{C}^{N^2-1})$ transformations (3.16) for generators of different representations of $SU(N)$:

$$(1.6) \quad L_G = \int \{ G_{ab} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}^a F_{\alpha\beta}^b + \bar{\Psi} [i\gamma^\mu (\partial_\mu + ig A_\mu^a \tau_a) - m] \Psi \} dx.$$

Here

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \lambda^{abc} A_\mu^b A_\nu^c.$$

6. It is very interesting that the unified field equations derived in [12] obey the PRI. In fact, with PRI, we are able to substantially reduce the to-be-determined parameters in our unified model to two $SU(2)$ and $SU(3)$ constant vectors

$$\{\alpha_\mu^w\} = (\alpha_1^w, \alpha_2^w, \alpha_3^w), \quad \{\alpha_k^s\} = (\alpha_1^s, \dots, \alpha_8^s),$$

containing 11 parameters as given in (4.31), representing the portions distributed to the gauge potentials by the weak and strong charges. Hence they are physically needed.

It appears that any field model with the classical Higgs scalar fields added to the action functional violates PRI, and hence can only be considered as an approximation for describing the related interactions. In fact, as far as we know, the unified field model introduced in [12] and refined in this article is the only model which obeys PRI. The main reason is that our model is derived from first principles, and the spontaneous gauge-symmetry breaking as well as the mechanism of mass generation and energy creation are natural outcomes of the constraint Lagrangian action (PID).

7. In the unified model, the coupling is achieved through PID in a transparent fashion, and consequently it can be easily decoupled. In other words, both PID and PRI can be applied directly to single interactions. For gravity, for example, we have derived modified Einstein equations, leading to a unified theory for dark matter and dark energy [10].

8. New gauge field equations for strong interaction, decoupled from the unified model, are derived by applying PID to the standard $SU(3)$ gauge action functional in QCD. The new model leads to consistent results as the classical QCD, and, more importantly, to a number of new results and predictions. In particular, this model gives rise to a natural duality between the $SU(3)$ gauge fields S_μ^k ($k = 1, \dots, 8$),

representing the gluons, and the adjoint scalar fields $\{\phi_s^k\}$, representing Higgs type of bosonic spin-0 particles.

9. One prediction from the duality from strong interaction is the existence of a Higgs type bosonic spin-0 particle with mass $m \geq 100\text{GeV}/c^2$. It is hoped that careful examination of the LHC data may verify the existence of this Higgs type of particle due to strong interaction.

10. For the first time, we derive three levels of strong interaction potentials: the quark potential S_q , the nucleon potential S_n and the atom/molecule potential S_a . They are given as follows:

$$(1.7) \quad S_q = g_s \left[\frac{1}{r} - \frac{Bk_0^2}{\rho_0} e^{-k_0 r} \varphi(r) \right],$$

$$(1.8) \quad S_n = 3 \left(\frac{\rho_0}{\rho_1} \right)^3 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_1} e^{-k_1 r} \varphi(r) \right],$$

$$(1.9) \quad S_a = 3N \left(\frac{\rho_0}{\rho_1} \right)^3 \left(\frac{\rho_1}{\rho_2} \right)^3 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_2} e^{-k_1 r} \varphi(r) \right],$$

where $\varphi(r) \sim r/2$, g_s is the strong charge, B, B_n are constants, $k_0 = mc/\hbar$, $k_1 = m_\pi c/\hbar$, m is mass of the above mentioned strong interaction Higgs particle, m_π is the mass of the Yukawa meson, ρ_0 is the effective quark radius, ρ_1 is the radius of a nucleon, ρ_2 is the radius of an atom/molecule, and N is the number of nucleons in an atom/molecule. These potentials match very well with experimental data, and offer a number of physical conclusions. Hereafter we shall explore a few important implications of these potentials.

11. With these strong interaction potentials, the binding energy of quarks can be estimated as

$$(1.10) \quad E_q \sim \left(\frac{\rho_1}{\rho_0} \right)^4 E_n \sim 10^{20} E_n,$$

where E_n is the binding energy of nucleons. Consequently, if the quark radius is considered as $\rho_0 \sim 10^{-21}\text{cm}$, then the Planck energy level 10^{19} GeV is required to break a quark free. Hence these potential formulas offer a clear mechanism for quark confinement.

12. With the quark potential, there is a radius \bar{r} , as shown in Figure 6.1, such that two quarks closer than \bar{r} are repelling, and for r near \bar{r} , the strong interaction diminishes. Hence this clearly explains asymptotic freedom.

13. In the nucleon level, the new potential is an improvement of the Yukawa potential. The corresponding Yukawa force is always attractive. However, as the distance between two nucleons is increasing, the nucleon force corresponding to the nucleon potential S_n behaves as repelling, then attracting, then repelling again and diminishes. This is exactly the picture that the observation tells us. In addition, these potentials give rise an estimate on the ratio between the gravitational and the strong interaction forces. This estimate indicates that near the radius of an atom, the strong repelling force is stronger than the gravitational force, and beyond the molecule radius, the strong repelling force is smaller than the gravitational force. We believe that it is this competition between the gravitational and the strong

forces in the level of atoms/molecules gives rise to the mechanism of the van der Waals force.

14. The factor $\left(\frac{\rho_0}{\rho_1}\right)^3 \left(\frac{\rho_1}{\rho_2}\right)^3$ in (1.9) indicates the strong interaction is of short-range, in agreement with observations. In particular, beyond molecular level, strong interaction diminishes. In addition, the derivation of these potentials clearly suggests that exchanging gluons leads to repelling force, and exchanging π -mesons (Higgs) leads to attracting force.

15. The new field equations for weak interaction, decoupled from the unified field model, provide a natural duality between weak gauge fields $\{W_\mu^a\}$, representing the W^\pm and Z intermediate vector bosons, and three bosonic scalar fields ϕ^a . A possible duality is the degenerate case where the three scalar fields ϕ^a are a constant vector ζ_a times a single scalar field ϕ , and the duality reduces to the duality between $\{W_\mu^a\}$ and one neutral Higgs boson field ϕ .

16. One key point of the study is that the field equations must satisfy PRI, which induces an important $SU(2)$ constant vector $\{\alpha_a^w\}$. The components of this vector represent the portions distributed to the gauge potentials W_μ^a by the weak charge g_w . Consequently, in the same spirit as electromagnetism, the time-components W_0^a of the gauge potentials represent the weak-charge potentials, and the total force exerted on a particle with N weak charges Ng_w is

$$(1.11) \quad F_{WE} = -Ng_w \alpha_a^w \nabla W_0^a.$$

It is the weak charge distribution vector α_a^w , due to PRI, that allows us to formulate the total weak potential/force as a field exerted on a particle. It is clear that F_{WE} is a representation invariant scalar, obeying PRI. This clearly overcomes one of the main difficulties encountered in classical theories.

In the same token, the spatial components $\vec{W}^a = (W_1^a, W_2^a, W_3^a)$ represent the weak-rotation potentials, yielding the following total weak-rotation force

$$(1.12) \quad F_{WM} = g_w \varepsilon^{abc} \alpha_a^w \vec{J}^b \times \text{curl} \vec{W}^c,$$

where $\{\vec{J}^b\} = \{J_1^b, J_2^b, J_3^b\}$ is the weak charge current density, and ε^{abc} is the structural constants using the Pauli matrices as generators for $SU(2)$. Also, F_{WM} is a representation invariant scalar, obeying PRI.

17. With the above physical meaning of the gauge potentials and the associated forces, for the first time, we derive the weak potential and weak force formula given by

$$(1.13) \quad \begin{aligned} W &= g_w e^{-k_1 r} \left[\frac{1}{r} - e^{-k_0 r} \psi(r) \right], \\ F &= g_w^2 e^{-k_1 r} \left[\frac{k_1}{r} + \frac{1}{r^2} - (K_1 \psi - \psi') e^{-k_0 r} \right], \end{aligned}$$

where $K_1 = k_0 + k_1$, $k_0 = m_H c / \hbar$, $k_1 = m_W c / \hbar$, m_H and m_W are the masses of the Higgs and W bosons, and $\psi(r) = \psi_1(r) + \psi_2(r) \ln r$ with $\psi_i(r)$ being polynomials; see (9.46). This force formula is consistent with observations: there is a radius $r_0 > 0$ such that F is repelling for $r < r_0$, and attractive for $r_0 < r < r_1$. In addition, F is a short-range force. Namely, F diminishes for $r \geq 10^{-16} \text{ cm}$.

18. With the duality, our analysis shows that the charged gauge bosons W^\pm do not appear simultaneously with the neutral boson Z in one physical situation. The same non-existence holds true for the neutral and charged Higgs particles as well.

19. The new duality model for weak interaction not only produces consistent physical conclusions as the classical GWS electroweak theory, but also leads to new insights and predictions for weak interaction. Here are a few similarities and distinctions between these two models:

- Both theories produces the right intermediate vector bosons W^\pm and Z , the neutral Higgs, the neutral current, and the scaling relation, consistent with experimental observations.
- The GWS model mixes transformations of different representations of $U(1)$ and $SU(2)$, and utilizes both the electromagnetic gauge potential and the weak gauge potentials to define the intermediate vector bosons. This gauge mixing causes the decoupling of the model to electromagnetic and weak components difficult, if not impossible. This mixing also violates PRI. The duality model used in this paper can be easily decoupled to study individual interactions involved, and satisfies PRI.
- In the GWS model, the Higgs mechanism of mass generation and energy creation is achieved by introducing the Higgs sector with a Higgs scalar field in the Lagrangian action functional. The mass generation and energy creation mechanism is achieved in a completely different and much simpler fashion in the duality model by using energy-momentum conservation constraint variation (PID) to the standard $SU(2)$ gauge functional.
- Due partially to mixing the gauge fields for electromagnetic and weak interactions, it is difficult to use the classical theory to derive any force/potential formulas for weak interaction. However, as mentioned earlier, the new duality model leads naturally to a long overdue force formula for weak interaction.

20. With both weak and strong *charge* potentials at our disposal, for the first time, we are able to introduce energy levels of leptons and quarks using W_μ , and energy levels for hadrons using S_μ . Then the standard conversion of the Dirac equation for a matter field leads to the following formulation of energy levels

$$(1.14) \quad -\nabla^2\Phi^w + \frac{g_w}{\hbar c}W_0(x)\Phi^w = \lambda^w\Phi^w \quad \text{for a lepton or a quark,}$$

$$(1.15) \quad -\nabla^2\Phi^H + \frac{g_s}{\hbar c}S_0(x)\Phi^H = \lambda^H\Phi^H \quad \text{for a hadron.}$$

We conclude then that each lepton or quark is represented by an eigenstate of (1.14) with corresponding eigenvalue being its binding energy, and the eigenstate of (1.14) with the lowest energy level represents the electron. Also, each hadron is represented by an eigenstate of (1.15) with the corresponding eigenvalue being its binding energy, and the eigenstate of (1.15) with the lowest energy level represents the proton.

21. A common feature of these force/charge potentials is that all four forces can be either repelling or attracting with different spatial scales.¹ This is the essence of the stability of matter in the universe from the smallest elementary particles to largest galaxies in the universe.

¹Attracting and repelling of electromagnetic force is achieved via the sign of the electric charge.

The paper is organized as follows. Section 2 recapitulates PID and its motivations. Section 3 introduces PRI, and the unified field models are refined in Section 4. Section 5 addresses the duality theory for different interactions. Sections 6 and 7 derive the strong interaction potentials and their implications. Section 8 addresses various features of the duality model for weak interaction. Section 9 derives the weak potential and force formulas, and Section 10 is devoted to the comparison between the classical GWS and the new electroweak theory. Section 11 recapitulates the weak and strong potentials, and Section 12 introduces energy levels of elementary particles. Section 13 offers our view on structure and stability of matter. Brief conclusions are given in Section 14.

This paper combines an early version of this article with [8, 9, 11].

2. MOTIVATIONS FOR PRINCIPLE OF INTERACTION DYNAMICS (PID)

2.1. Recapitulation of PID. We first recall the principle of interaction dynamics (PID) proposed in [12]. Let (M, g_{ij}) be the 4-dimensional space-time Riemannian manifold with $\{g_{ij}\}$ the Minkowski type Riemannian metric. For an (r, s) -tensor u we define the A -gradient and A -divergence operators ∇_A and div_A as

$$(2.1) \quad \begin{aligned} \nabla_A u &= \nabla u + u \otimes A, \\ \text{div}_A u &= \text{div} u - A \cdot u, \end{aligned}$$

where A is a vector or co-vector field, ∇ and div are the usual gradient and divergent covariant differential operators.

Let $F = F(u)$ be a functional of a tensor field u . A tensor u_0 is called an extremum point of F with the div_A -free constraint, if

$$(2.2) \quad \left. \frac{d}{d\lambda} F(u_0 + \lambda X) \right|_{\lambda=0} = \int_M \delta F(u_0) \cdot X \sqrt{-g} dx = 0 \quad \forall \text{div}_A X = 0.$$

We now state PID, first introduced by the authors in [12].

Principle of Interaction Dynamics (PID). *For all physical interactions there are Lagrangian actions*

$$(2.3) \quad L(g, A, \psi) = \int_M \mathcal{L}(g_{ij}, A, \psi) \sqrt{-g} dx,$$

where $g = \{g_{ij}\}$ is the Riemann metric representing the gravitational potential, A is a set of vector fields representing gauge and mass potentials, and ψ are the wave functions of particles. The action (2.3) satisfy the invariance of general relativity (or Lorentz invariance), the gauge invariance, and PID. Moreover, the states (g, A, ψ) are the extremum points of (2.3) with the div_A -free constraint (2.2).

The following theorem is crucial for applications of PID.

Theorem 2.1 (Ma and Wang [10, 12]). *Let $F = F(g_{ij}, A)$ be a functional of Riemannian metric $\{g_{ij}\}$ and vector fields A^1, \dots, A^N . For the div_A -free constraint variations of F , we have the following assertions:*

- (1) *There is a vector field $\Phi \in H^1(TM)$ such that the extremum points $\{g_{ij}\}$ of F with the div_A -free constraint satisfy the equations*

$$(2.4) \quad \frac{\delta}{\delta g_{ij}} F(g_{ij}) = \left(\nabla_i + \sum_{k=1}^N \alpha_k A_i^k \right) \Phi_j$$

where α_k ($1 \leq k \leq N$) are parameters, $\nabla_i \Phi_j = \partial_i \Phi_j - \Gamma_{ij}^l \Phi_l$ are the covariant derivatives, and $\text{div}_A X = \text{div} X - \sum_{k=1}^N \alpha_k A^k \cdot X$.

- (2) If the first Betti number of M is zero, and $A^k = 0$ ($1 \leq k \leq N$) in (2.4), then there exists a scalar field φ such that $\Phi = \nabla \varphi$, i.e. equations (2.4) become

$$(2.5) \quad \frac{\delta}{\delta g_{ij}} F(g_{ij}) = -\nabla_i \nabla_j \varphi.$$

- (3) For each A^a , there is a scalar function $\varphi^a \in H^1(M)$ such that the extremum points A^a of F with the div_A -free constraint satisfy the equations

$$(2.6) \quad \frac{\delta}{\delta A_\mu^a} F(A^a) = (\nabla_\mu + \beta_b^a A_\mu^b) \varphi^a$$

where β_b^a are parameters, $\text{div}_A X^a = \text{div} X^a - \beta_b^a A_\mu^b X_\mu^a$ for the a -th vector field X^a .

Based on PID and Theorem 2.1, the field equations with respect to the action (2.3) are given in the form

$$(2.7) \quad \frac{\delta}{\delta g_{\mu\nu}} L(g, A, \psi) = (\nabla_\mu + \alpha_b A_\mu^b) \Phi_\nu,$$

$$(2.8) \quad \frac{\delta}{\delta A_\mu^a} L(g, A, \psi) = (\nabla_\mu + \beta_b^a A_\mu^b) \varphi^a,$$

$$(2.9) \quad \frac{\delta}{\delta \psi} L(g, A, \psi) = 0,$$

where $A_\mu^k = (A_0^k, A_1^k, A_2^k, A_3^k)$ ($1 \leq k \leq N, N = 12$) are the gauge vector fields for the electromagnetic, weak, and strong interactions, $\Phi_\mu = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ is a vector field induced by gravitational interaction, φ^a are scalar fields generated from the gauge field A^a , and α_b, β_b ($1 \leq b \leq N$) are coupling parameters.

Consider the action (2.3) as the natural combination of the four actions

$$\mathcal{L} = \mathcal{L}_{HE} + \mathcal{L}_{QED} + \mathcal{L}_W + \mathcal{L}_{QCD},$$

where \mathcal{L}_{HE} is the Einstein-Hilbert action, \mathcal{L}_{QED} is the QED gauge action, \mathcal{L}_W is the $SU(2)$ gauge actions for weak interactions, and \mathcal{L}_{QCD} is the action for the quantum chromodynamics. Then (2.7)-(2.9) provide the unified field equations coupling all interactions. Moreover, we see from (2.7)-(2.9) that there are too many coupling parameters need to be determined. Fortunately, this problem can be satisfactorily resolved, leading also the discovery of PRI.

In the remaining parts of this sections, we shall give some evidences and motivations for PID.

2.2. Dark matter and dark energy. The presence of dark matter and dark energy provides a strong support for PID. In this case, the energy-momentum tensor T_{ij} of normal matter is no longer conserved:

$$\nabla^i (T_{ij}) \neq 0.$$

Then as mentioned in the Introduction and in [12], the gravitational field equations (1.1) are uniquely determined by constraint Lagrangian variation, i.e. by PID. Also,

by (1.2), the added term $\nabla_i \nabla_j \varphi$ has no variational structure. In other words, the term $\Phi = g^{ij} \nabla_i \nabla_j \varphi$ cannot be added into the Einstein-Hilbert functional.

2.3. Higgs mechanism and mass generation. Higgs mechanism is another main motivation to postulate PID in our program for a unified field theory. In the Glashow-Weinberg-Salam (GWS) electroweak theory, the three force intermediate vector bosons W^\pm, Z for weak interaction retain their masses by spontaneous gauge-symmetry breaking, which is called the Higgs mechanism. We now show that the masses of the intermediate vector bosons can be also attained by PID. In fact, we shall further show in Section 4 that all conclusions of the GWS electroweak theory confirmed by experiments can be derived by the electroweak theory based on PID.

For convenience, we first introduce some related basic knowledge on quantum physics. In quantum field theory, a field ψ is called a fermion with mass m , if it satisfies the Dirac equation

$$(2.10) \quad (i\gamma^\mu \partial_\mu - m)\psi = 0,$$

where γ^μ are the Dirac matrices. The action of (2.10) is

$$(2.11) \quad L_F = \int \mathcal{L}_F dx, \quad \mathcal{L}_F = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

A field Φ is called a boson with mass m , if Φ satisfies the Klein-Gordon equation

$$(2.12) \quad \square \Phi + \left(\frac{mc}{\hbar}\right)^2 \Phi = o(\Phi),$$

where $o(\Phi)$ is the higher order terms of Φ , and \square is the wave operator given by

$$\square = \partial^\mu \partial_\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2.$$

The bosonic field Φ is massless if it satisfies

$$(2.13) \quad \square \Phi = o(\Phi).$$

The physical significances of the fermion and bosonic fields ψ and Φ are as follows:

- (1) Macro-scale: Ψ, Φ represent field energy.
- (2) Micro-sale (i.e. Quantization): ψ represents a spin- $\frac{1}{2}$ fermion (particle), and Φ represents a bosonic particle with an integer spin k if Φ is a k -tensor field.

In particular, in the classical Yang-Mills theory, a gauge field $\{A_\mu\} = (A_0, A_1, A_2, A_3)$ satisfies the following field equations:

$$(2.14) \quad \partial^\mu F_{\mu\nu} = o(A), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

which are the Euler-Lagrange of the Yang-Mills action

$$(2.15) \quad L_{YM} = \int (F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_F + o(A)) dx$$

where \mathcal{L}_F is as in (2.11), and

$$\partial^\mu F_{\mu\nu} = \square A_\nu - \partial_\nu (\partial^\mu A_\mu)$$

Thus, for a fixed gauge

$$(2.16) \quad \text{div} A = \partial^\mu A_\mu = \text{constant},$$

the gauge field equations (2.14) are reduced to the bosonic field equations (2.13). In other words, the gauge field A satisfying (2.14) is a spin-1 massless boson, as A is a vector field.

We are now in position to introduce the Higgs mechanism. Physical experiments show that weak interacting fields should be gauge fields with masses. However, as mentioned in (2.14), the gauge fields satisfying Yang-Mills theory are massless. In this situation, the six physicists, Higgs [5], Englert and Brout [2], Guralnik, Hagen and Kibble [4], suggested to add a scalar field ϕ into the Yang-Mills functional (2.15) to create masses.

For clearly revealing the essence of the Higgs mechanism, we only take one gauge field (there are four gauge fields in the GWS theory). In this case, the Yang-Mills action density is in the form

$$(2.17) \quad \mathcal{L}_{YM} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi,$$

where $g^{\mu\nu}$ is the Minkowski metric,

$$(2.18) \quad D_\mu \psi = (\partial_\mu + igA_\mu)\psi,$$

and g is a constant. It is clear that (2.17)-(2.18) are invariant under the following $U(1)$ gauge transformation

$$(2.19) \quad \psi \rightarrow e^{i\theta}\psi, \quad A_\mu \rightarrow A_\mu - \frac{1}{g}\partial_\mu\theta.$$

The Euler-Lagrange equations of (2.17) are

$$(2.20) \quad \begin{aligned} \square A_\mu - \partial_\mu(\text{div} A) - gJ_\mu &= 0, \\ (i\gamma^\mu D_\mu - m)\psi &= 0, \\ J_\mu &= i\bar{\psi}\gamma^\mu\psi, \end{aligned}$$

which are invariant under the gauge transformation (2.19). In (2.20) the bosonic particle A_μ is massless.

Now, we add a Higgs action \mathcal{L}_H into (2.17):

$$(2.21) \quad \begin{aligned} \mathcal{L}_H &= \frac{1}{2}g^{\mu\nu}(D_\mu\phi)^\dagger(D_\nu\phi) - \frac{1}{4}(\phi^\dagger\phi - \rho)^2, \\ D_\mu\phi &= (\partial_\mu + igA_\mu)\phi, \\ (D_\mu\phi)^\dagger &= (\partial_\mu - igA_\mu)\phi^\dagger, \end{aligned}$$

where $\rho \neq 0$ is a constant. Obviously, the following action and its variational equations

$$(2.22) \quad L = \int (\mathcal{L}_{YM} + \mathcal{L}_H) dx,$$

$$(2.23) \quad \begin{cases} \frac{\delta L}{\delta A^\mu} = \partial^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu) - gJ_\mu + \frac{ig}{2}(\phi(D_\mu\phi)^\dagger - \phi^\dagger D_\mu\phi) = 0, \\ \frac{\partial L}{\delta \psi} = (i\gamma^\mu D_\mu - m)\psi = 0, \\ -\frac{\delta L}{\delta \phi^*} = (D^\mu)^\dagger D_\mu\phi + (\phi^\dagger\phi - \rho^2)\phi = 0, \end{cases}$$

are invariant under the gauge transformation

$$(2.24) \quad (\psi, \phi) \rightarrow (e^{i\theta}\psi, e^{i\theta}\phi), \quad A_\mu \rightarrow A_\mu - \frac{1}{g}\partial_\mu\theta.$$

The equations (2.22) are still massless. However, we note that $(0, 0, \rho)$ is a solution of (2.23), which is a ground state, i.e. a vacuum state. Consider a translation for $\Phi = (A, \psi, \phi)$ at $\Phi_0 = (0, 0, \rho)$ as

$$(2.25) \quad \Phi = \tilde{\Phi} + \Phi_0, \quad \tilde{\Phi} = (\tilde{A}, \tilde{\psi}, \tilde{\phi}),$$

then the equations (2.23) become

$$(2.26) \quad \begin{aligned} \partial^\nu(\partial_\nu\tilde{A}_\mu - \partial_\mu\tilde{A}_\nu) + g\rho\tilde{A}_\mu - g\tilde{J}_\mu + \frac{ig}{2}\tilde{J}_\nu(\tilde{\phi}) &= 0, \\ (i\gamma^\mu D_\mu - m)\tilde{\psi} &= 0, \\ (D^\mu)^\dagger D_\mu(\tilde{\phi} + \rho) + ((\tilde{\phi} + \rho)^\dagger(\tilde{\phi}_\rho) - \rho^2)(\tilde{\phi} - \rho) &= 0, \end{aligned}$$

where

$$J_\mu(\tilde{\phi}) = \tilde{\phi}(D_\mu\tilde{\phi})^\dagger - \tilde{\phi}^\dagger D_\mu\tilde{\phi}.$$

We see that \tilde{A}_μ obtains its mass $m = \sqrt{g\rho}$ in (2.26). Equations (2.26) break the invariance for the gauge transformation (2.24), and masses are created by the spontaneous gauge-symmetry breaking, called the Higgs mechanism, and $\tilde{\phi}$ is the Higgs boson.

In the following, we show that PID provides a new mechanism of creating masses, very different from the Higgs mechanism.

In view of the equations (2.8)-(2.9) based on PID, the variational equations of the Yang-Mills action (2.17) with the div_A -free constraint are in the form

$$(2.27) \quad \begin{aligned} \partial^\nu F_{\nu\mu} - gJ_\mu &= \left[\partial_\mu + \frac{1}{4} \left(\frac{mc}{\hbar} \right)^2 x_\mu - \lambda A_\mu \right] \phi, \\ (i\gamma^\mu D_\mu - m_f)\psi &= 0, \end{aligned}$$

where ϕ is a scalar field, $\frac{1}{4} \left(\frac{mc}{\hbar} \right)^2 x_\mu$ is the mass potential of the scalar field ϕ , and $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$. If ϕ has a nonzero ground state $\phi_0 = \rho$, then for the translation

$$\phi = \tilde{\phi} + \rho, \quad A_\mu = \tilde{A}_\mu, \quad \psi = \tilde{\psi},$$

the first equation of (2.27) becomes

$$(2.28) \quad \partial^\nu \tilde{F}_{\nu\mu} + \left(\frac{m_0 c}{\hbar} \right)^2 \tilde{A}_\mu - g\tilde{J}_\mu = \left[\partial_\mu + \frac{1}{4} \left(\frac{mc}{\hbar} \right)^2 x_\mu - \lambda \tilde{A}_\mu \right] \tilde{\phi},$$

where $\left(\frac{m_0 c}{\hbar} \right)^2 = \lambda\rho$. Thus the mass $m_0 = \frac{\hbar}{c}\sqrt{\lambda\rho}$ is created in (2.28) as the Yang-Mills action takes the div_A -free constraint variation. When we take divergence on both sides of (2.28), and by

$$\partial^\mu \partial^\nu \tilde{F}_{\nu\mu} = 0, \quad \partial^\mu \tilde{J}_\mu = 0,$$

we derive that the field equation of $\tilde{\phi}$ are given by

$$(2.29) \quad \partial^\mu \partial_\nu \tilde{\phi} + \left(\frac{mc}{\hbar} \right)^2 \tilde{\phi} = \lambda A_\mu \partial^\mu \tilde{\phi} - \left(\frac{mc}{\hbar} \right)^2 x_\mu \partial^\mu \tilde{\phi}.$$

The equation (2.29) corresponds to the Higgs field equation, the third equation in (2.26), with a fixed gauge

$$\operatorname{div} \tilde{A} = \frac{\rho}{\lambda} \left(\frac{m_0 c}{\hbar} \right)^2 = \rho^2,$$

and the value m_0 is the mass of the bosonic particle $\tilde{\phi}$. Here we remark that the essence of the Higgs mechanism is to add an action ad hoc. However, for the field model with PID, the mass is generated naturally.

2.4. Ginzburg-Landau superconductivity. Superconductivity studies the behaviors of the Bose-Einstein condensation and electromagnetic interactions. The Ginzburg-Landau theory provides a support for PID.

The Ginzburg-Landau free energy for superconductivity is given by

$$(2.30) \quad G = \int_{\Omega} \left[\frac{1}{2M_s} |(ih\nabla + \frac{e_s}{c}A)\psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{8\pi} |\operatorname{curl} A|^2 \right] dx,$$

where A is the electromagnetic potential, ψ is the wave function of superconducting electrons, Ω is the superconductor, e_s and m_s are the charge and mass of a Cooper pair.

The superconducting current equations determined by the Ginzburg-Landau free energy (2.30) are:

$$(2.31) \quad \frac{\delta G}{\delta A} = 0,$$

which implies that

$$(2.32) \quad \frac{c}{4\pi} \operatorname{curl}^2 A = -\frac{e_s^2}{m_s c} |\psi|^2 A - i \frac{\hbar e_s}{m_s} (\psi^* \nabla \psi - \psi \nabla \psi^*).$$

Let

$$J = \frac{c}{4\pi} \operatorname{curl}^2 A, \quad J_s = -\frac{e_s^2}{m_s c} |\psi|^2 A - i \frac{\hbar e_s}{m_s} (\psi^* \nabla \psi - \psi \nabla \psi^*).$$

Physically, J is the total current in Ω , and J_s is the superconducting current. Since Ω is a medium conductor, J contains two types of currents

$$J = J_s + \sigma E,$$

where σE is the current generated by electric field E ,

$$E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \Phi = -\nabla \Phi,$$

Φ is the electric potential. Therefore, the superconducting current equations should be taken as

$$(2.33) \quad \frac{1}{4\pi} \operatorname{curl} A = -\frac{\sigma}{c} \nabla \Phi - \frac{e_s^2}{m_s c^2} |\psi|^2 A - \frac{i \hbar e_s}{m_s c} (\psi^* \nabla \psi - \psi \nabla \psi^*).$$

Comparing with (2.31) and (2.32), we find that the equations (2.33) are in the form

$$(2.34) \quad \frac{\delta G}{\delta A} = -\frac{\sigma}{c} \nabla \Phi.$$

In addition, for conductivity the fixing gauge is

$$\operatorname{div} A = 0, \quad A \cdot n|_{\partial\Omega} = 0,$$

which implies that

$$\int_{\Omega} \nabla \Phi \cdot A dx = 0.$$

Hence the term $-\frac{\sigma}{c} \nabla \Phi$ in (2.34) can not be added into the Ginzburg-Landau free energy (2.30).

However, the equations (2.34) are just the div-free constraint variational equations:

$$\left(\frac{\delta G}{\delta A}, B \right) = \frac{d}{d\lambda} G(A + \lambda B) \Big|_{\lambda=0} = 0 \quad \forall \operatorname{div} B = 0.$$

Thus, we see PID is valid for the Ginzburg-Landau superconductivity theory.

3. PRINCIPLE OF REPRESENTATION INVARIANCE (PRI)

3.1. Yang-Mills gauge fields. In this section, we present a new symmetry for gauge field theory, called the (gauge group) representation invariance. To this end we first recall briefly the Yang-Mills gauge field theory.

The simplest gauge field is a vector field A_μ and a Dirac spinor field ψ (also called fermion field):

$$A_\mu = (A_0, A_1, A_2, A_3)^T \quad \text{and} \quad \psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T,$$

such that the action (2.17) with (2.18) is invariant under the $U(1)$ gauge transformation (2.19). The electromagnetic interaction is described by a $U(1)$ gauge field.

In the general case, a set of $SU(N)$ ($N \geq 2$) gauge fields consists of $K = N^2 - 1$ vector fields A_μ^a and N spinor fields ψ^j :

$$(3.1) \quad A_\mu^1, \dots, A_\mu^K, \quad \Psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix}, \quad \psi^j = \begin{pmatrix} \psi_1^j \\ \psi_2^j \\ \psi_3^j \\ \psi_4^j \end{pmatrix},$$

which have to satisfy the $SU(N)$ gauge invariance defined as follows.

First, the N spinor fields Ψ in (3.1) describe N fermions, satisfying the Dirac equations

$$(3.2) \quad i\gamma^\mu D_\mu \Psi - m\Psi = 0,$$

where the mass matrix m and the derivative operators D_μ are defined by

$$(3.3) \quad m = \begin{pmatrix} m_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_N \end{pmatrix}, \quad D_\mu = \partial_\mu + igA_\mu^a \tau^a,$$

where A_μ^a ($1 \leq a \leq k$) are vector fields given by (3.1), and τ^a are $K = N^2 - 1$ given complex matrices as

$$\tau^a = \begin{pmatrix} z_{11}^a & \cdots & z_{1N}^a \\ \vdots & \ddots & \vdots \\ z_{N1}^a & \cdots & z_{NN}^a \end{pmatrix} \quad \forall 1 \leq a \leq K = N^2 - 1,$$

which satisfies

$$(3.4) \quad \tau^a = \tau^{a\dagger}, \quad [\tau^a, \tau^b] = i\lambda^{abc} \tau^c,$$

where $[\tau^a, \tau^b] = \tau^a \tau^b - \tau^b \tau^a$, and λ^{abc} are the structural constants of $SU(N)$.

The reason that D_μ in (3.2) take the form (3.3)-(3.4) is that certain physical properties of the N fermions ψ^1, \dots, ψ^N are not distinguishable under the $SU(N)$ transformations:

$$(3.5) \quad \tilde{\Psi}(x) = U(x)\Psi(x), \quad U(x) \in SU(N) \quad \forall x \in M,$$

where M is the Minkowski space-time manifold. Consequently, it requires that the Dirac equations (3.2) are covariant under the $SU(N)$ transformation (3.5).

On the other hand, each element $U \in SU(N)$ can be expressed as

$$U = e^{i\theta^a \tau^a},$$

where τ^a is as in (3.4), and θ^a ($1 \leq a \leq N^2 - 1$) are real parameters. Therefore (3.5) can be written as

$$(3.6) \quad \tilde{\Psi}(x) = e^{i\theta^a(x)\tau^a} \Psi(x).$$

The covariance of (3.2) implies that

$$(3.7) \quad \tilde{D}_\mu \tilde{\Psi} = U(x) D_\mu \Psi, \quad U(x) = e^{i\theta^a(x)\tau^a}.$$

Namely,

$$\begin{aligned} \tilde{D}_\mu \tilde{\Psi} &= \partial_\mu \tilde{\Psi} + ig \tilde{A}_\mu^a \tau^a \tilde{\Psi} \\ &= U \partial_\mu \Psi + (\partial_\mu U) \Psi + ig \tilde{A}_\mu^a \tau^a U \Psi \\ &= U [\partial_\mu \Psi + ig A_\mu^a \tau^a \Psi], \end{aligned}$$

from which we obtain the transformation rule for A_μ^a and the mass matrix m defined by (3.3), ensuring the covariance (3.7):

$$(3.8) \quad \begin{aligned} \tilde{A}_\mu^a \tau^a &= \frac{i}{g} (\partial_\mu U) \Psi + U A_\mu^a \tau^a U^{-1}, \\ \tilde{m} &= U m U^{-1}. \end{aligned}$$

Thus under the $SU(N)$ gauge transformation (3.8), equations (3.2) are covariant. Now we need to find the equations for A_μ^a obeying the covariance under the gauge transformation (3.6) and (3.8). Since D_μ in (3.3) satisfy (3.7) and by (3.4), the commutator

$$\begin{aligned} \frac{i}{g} [D_\mu, D_\nu] &= \frac{i}{g} (\partial_\mu + ig A_\mu^a \tau^a) (\partial_\nu + ig A_\nu^a \tau^a) - \frac{i}{g} (\partial_\nu + ig A_\nu^a \tau^a) (\partial_\mu + ig A_\mu^a \tau^a) \\ &= \partial_\mu A_\nu^a \tau^a - \partial_\nu A_\mu^a \tau^a - ig [A_\mu^a \tau^a, A_\nu^a \tau^a] \\ &= (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \lambda^{abc} A_\mu^b A_\nu^c) \tau^a \end{aligned}$$

has the covariance:

$$[\tilde{D}_\mu, \tilde{D}_\nu] = U [D_\mu, D_\nu] U^\dagger, \quad (U^\dagger = U^{-1}).$$

Hence defining

$$F_{\mu\nu} = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \lambda^{abc} A_\mu^b A_\nu^c) \tau^a,$$

we derive the invariance

$$\text{Tr}(\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}) = \text{Tr}(U F_{\mu\nu} U^{-1} U F^{\mu\nu} U^{-1}) = \text{Tr}(F_{\mu\nu} F^{\mu\nu}) = F_{\mu\nu}^a F^{\mu\nu a}.$$

where

$$(3.9) \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \lambda^{abc} A_\mu^b A_\nu^c,$$

Thus the functional of the gauge field A

$$(3.10) \quad L = \int F_{\mu\nu}^a F^{\mu\nu a} dx,$$

is invariant, and the Euler-Lagrange equations of (3.10) are covariant under the gauge transformation (3.8).

3.2. $SU(N)$ tensors. We now know that the $SU(N)$ gauge fields have $K = N^2 - 1$ vector fields A_μ^a ($1 \leq a \leq K$) and N fermion wave functions Ψ :

$$(3.11) \quad A_\mu^a = \begin{pmatrix} A_\mu^1 \\ \vdots \\ A_\mu^K \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix},$$

such that the action

$$(3.12) \quad L = \int (\mathcal{L}_G + \mathcal{L}_F) dx,$$

is invariant under the gauge transformation (3.6) and (3.8), which can be equivalently rewritten for infinitesimal θ^a as

$$(3.13) \quad \begin{aligned} \tilde{\Psi} &= e^{i\theta^a \tau^a} \Psi, \\ \tilde{A}_\mu^a &= A_\mu^a - \frac{1}{g} \partial_\mu \theta^a + \lambda^{abc} \theta^b A_\mu^c, \end{aligned}$$

where τ^a is as in (3.4), $\mathcal{L}_G, \mathcal{L}_F$ are the gauge and fermion action densities given by

$$(3.14) \quad \begin{aligned} \mathcal{L}_G &= g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}^a F_{\alpha\beta}^a, \\ \mathcal{L}_F &= \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi, \end{aligned}$$

where $F_{\mu\nu}^a$ is as in (3.9), $g^{\mu\nu}$ is the Minkowski metric, m and D_μ are as in (3.2) and (3.3), and

$$\bar{\Psi} = (\bar{\psi}^1, \dots, \bar{\psi}^N), \quad \bar{\psi}^k = \psi^{k\dagger} \gamma^0, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

For the above gauge field theory, a very important problem is that there are infinite number of families of generators

$$\{\tau^a \mid 1 \leq a \leq K = N^2 - 1\}$$

of $SU(N)$, and each family of generators $\{\tau^a\}$ corresponds to a group of gauge fields $\{A_\mu^a\}$:

$$(3.15) \quad \{\tau^a \mid 1 \leq a \leq K\} \longleftrightarrow \{A_\mu^a \mid 1 \leq a \leq K\}.$$

Intuitively, any gauge theory should be independent of the choice of $\{\tau^a\}$. However, the Yang-Mills functional (3.12) violates this principle, i.e. the form of (3.12) will change under the gauge transformation

$$A_\mu^a \rightarrow x_b^a A_\mu^b,$$

where (x_b^a) is a $K \times K$ nondegenerate complex matrix.

To solve this problem, we need to establish a new gauge invariance theory. Hence we introduce the $SU(N)$ tensors.

In mathematics, $SU(N)$ is an $N^2 - 1$ dimensional manifold, and the tangent space of $SU(N)$ at the unit element $e = I$ is characterized as

$$T_e SU(N) = \{i\tau \in M(\mathbb{C}^N) \mid \tau = \tau^\dagger\},$$

where $M(\mathbb{C}^N)$ is the linear space of all $N \times N$ complex matrices, and $T_e SU(N)$ is an $N^2 - 1$ -dimensional real linear space. Hence, each generator $\{\tau^1, \dots, \tau^K\}$ of $SU(N)$ can be regarded as a basis of $T_e SU(N)$. For consistency with the notations of tensors, we denote

$$\tau_a = \{\tau_1, \dots, \tau_K\} \subset T_e SU(N)$$

as a basis of $T_e SU(N)$. Take a basis transformation

$$(3.16) \quad \tilde{\tau}_a = x_a^b \tau_b \quad (\text{or } \tilde{\tau} = X\tau),$$

where $X = (x_a^b)$ is a nondegenerate complex matrix, and denote the inverse of X by $X^{-1} = (\tilde{x}_a^b)$. Under the transformation (3.16), the coordinate $\theta^a = (\theta^1, \dots, \theta^K)$ corresponding to the basis τ_a and the gauge field A_μ^a as (3.15) will transform as follows

$$(3.17) \quad \tilde{\theta}^a = \tilde{x}_b^a \theta^b, \quad \tilde{A}_\mu^a = \tilde{x}_b^a A_\mu^b.$$

In addition, we note that

$$[\tau_a, \tau_b] = i\lambda_{ab}^c \tau_c,$$

where λ_{ab}^c are the structural constants. By (3.16) we have

$$\begin{aligned} [\tilde{\tau}_a, \tilde{\tau}_b] &= i\tilde{\lambda}_{ab}^c \tilde{\tau}_c = i\tilde{\lambda}_{ab}^c x_c^d \tau_d, \\ [\tilde{\tau}_a, \tilde{\tau}_b] &= x_d^c x_b^d [\tau_c, \tau_d] = i x_a^c x_b^d \lambda_{cd}^f \tau_f. \end{aligned}$$

It follows that

$$(3.18) \quad \tilde{\lambda}_{ab}^c = x_a^f x_b^g \tilde{x}_d^c \lambda_{fg}^d.$$

From (3.17) and (3.18) we see that θ^a, A_μ^a transform in the form of vector fields, and the structural constants λ_{ab}^c transform as (1,2)-tensors. Thus, the quantities θ^a, A_μ^a , and λ_{ab}^c are called $SU(N)$ -tensors, which are crucial for introducing an invariance theory for the gauge fields.

From the structural constants λ_{ab}^c , we can construct an important $SU(N)$ -tensor G_{ab} , which can be regarded as a Riemannian metric defined on $SU(N)$. In fact, G_{ab} is a 2nd-order covariant $SU(N)$ -tensor given by

$$(3.19) \quad G_{ab} = \frac{1}{4N} \lambda_{ad}^c \lambda_{cb}^d.$$

3.3. Principle of Representation Invariance. As mentioned above, a physically sound gauge theory should be invariant under the $SU(N)$ representation transformation (3.16). In the same spirit as the Einstein's principle of relativity, we postulate the following principle of representation invariance (PRI).

Principle of Representation Invariance (PRI). *All $SU(N)$ gauge theories are invariant under the transformation (3.16). Namely, the actions of the gauge fields are invariant and the corresponding gauge field equations are covariant under the transformation (3.16).*

It is easy to see that the classical Yang-Mills actions (3.12) violate the PRI for the general representation transformations (3.16)-(3.18). The modified invariant actions should be in the form

$$(3.20) \quad \begin{aligned} L_G &= \int [\mathcal{L}_G + \mathcal{L}_F] dx, \\ \mathcal{L}_G &= G_{ab} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}^a F_{\alpha\beta}^b, \\ \mathcal{L} &= \bar{\Psi} [i\gamma^\mu (\partial_\mu + ig A_\mu^a \tau_a) - m] \Psi, \end{aligned}$$

where G_{ab} is defined as in (3.19).

To ensure that the action (3.20) is well-defined, the matrix (G_{ab}) must be symmetric and positive definite. In fact, by $\lambda_{ab}^c = -\lambda_{ba}^c$ we have

$$G_{ab} = \lambda_{ad}^c \lambda_{cb}^d = \lambda_{da}^c \lambda_{bc}^d = G_{ba}.$$

Hence (G_{ab}) is symmetric. The positivity of (G_{ab}) can be proved if (G_{ab}) is positive for a given generator τ_a of $SU(N)$. In the following we show that both $SU(2)$ and $SU(3)$, two most important cases in physics, possess positive matrices (G_{ab}) .

To see this, first consider $SU(2)$. We take the Pauli matrices

$$(3.21) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a given family of generators of $SU(2)$. The corresponding structural constants are given by

$$\lambda_{ab}^c = 2\varepsilon_{abc}, \quad \varepsilon_{abc} = \begin{cases} 1 & \text{if } (abc) \text{ is an even permutation of } (123), \\ -1 & \text{if } (abc) \text{ is an odd permutation of } (123), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$G_{ab} = \frac{1}{8} \lambda_{ab}^c \lambda_{ca}^b = \delta_{ab}.$$

Namely (G_{ab}) is an Euclidian metric. Thus for the representation with generators (3.21), the action (3.20) is the same as the classical Yang-Mills.

Second, for $SU(3)$, we take the generators of the Gell-Mann representation of $SU(3)$ as

$$(3.22) \quad \begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

The structural constants are

$$\lambda_{ab}^c = 2f_{abc}, \quad 1 \leq a, b, c \leq 8,$$

where f_{abc} are antisymmetric, and

$$(3.23) \quad \begin{aligned} f_{123} &= 1, & f_{147} &= -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}, \\ f_{458} &= f_{678} = \frac{\sqrt{3}}{2}. \end{aligned}$$

We infer from (3.23) that

$$(3.24) \quad \begin{aligned} \lambda_{ad}^c \lambda_{cb}^d &= 0 \quad \forall a \neq b, \\ \lambda_{ab}^c \lambda_{ca}^b &= 12 \quad \forall 1 \leq a \leq 8. \end{aligned}$$

Hence we have

$$G_{ab} = \frac{1}{12} \lambda_{ad}^c \lambda_{cb}^d = \delta_{ab}.$$

Again, (G_{ab}) is an Euclid metric for the Gell-Mann representation (3.22) of $SU(3)$.

In fact, for all $N \geq 2$ there exists a representation $\{\tau_a\}$ of $SU(N)$ generators, such that the metric $G_{ab} = \delta_{ab}$ is Euclidian. These $N \times N$ matrices τ_a can be taken in the form

$$(3.25) \quad \begin{aligned} \tau_1^{(1)} &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}, & \tau_2^{(1)} &= \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, & \tau_3^{(1)} &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \\ \tau_1^{(2)} &= \begin{pmatrix} \lambda_4 & 0 \\ 0 & 0 \end{pmatrix}, & \tau_2^{(2)} &= \begin{pmatrix} \lambda_5 & 0 \\ 0 & 0 \end{pmatrix}, & \tau_3^{(2)} &= \begin{pmatrix} \lambda_6 & 0 \\ 0 & 0 \end{pmatrix}, \\ \tau_4^{(2)} &= \begin{pmatrix} \lambda_7 & 0 \\ 0 & 0 \end{pmatrix}, & \tau_5^{(2)} &= \begin{pmatrix} \lambda_8 & 0 \\ 0 & 0 \end{pmatrix}, \\ & \vdots & & \\ \tau_1^{(N-1)} &= \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, & \tau_2^{(N-1)} &= \begin{pmatrix} 0 & \cdots & -i \\ \vdots & \ddots & \vdots \\ i & \cdots & 0 \end{pmatrix}, \\ \tau_3^{(N-1)} &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}, & \tau_4 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -i \\ \vdots & \vdots & & \vdots \\ 0 & i & \cdots & 0 \end{pmatrix}, \\ & \vdots & & \\ \tau_{2N-1}^{(N-1)} &= \begin{pmatrix} \text{Id} & 0 \\ 0 & -(N-1) \end{pmatrix}, \end{aligned}$$

where σ_i ($1 \leq i \leq 3$) and λ_k ($4 \leq k \leq 8$) are as in (3.21) and (3.22). With these generators (3.25),

$$(3.26) \quad G_{ab} = \frac{1}{4N} \lambda_{ad}^c \lambda_{cb}^d = \delta_{ab}.$$

Thus the 2nd-order covariant gauge tensor $\{G_{ab}\}$ is symmetric and positive definite, and defines a Riemannian metric on $SU(N)$ by taking the inner product in $T_B SU(N)$ as

$$\langle d\theta^a, d\theta^b \rangle = G_{ab}(B) d\theta^a d\theta^b \quad \forall B \in SU(N).$$

3.4. Unitary rotation gauge invariance. In the above subsection, we have proposed the PRI, and established a covariant theory for the $SU(N)$ gauge fields (3.11) under a general basis transformation (3.16). In (3.26) we see that the $SU(N)$ tensor $\{G_{ab}\}$ gives rise to an Euclidian metric if we take τ_a as in (3.25). We know that the same linear combinations of gauge fields

$$\tilde{A}_\mu^a = z_b^a A_\mu^b, \quad z_b^a \in \mathbb{C},$$

represent interacting field particles provided the matrix (z_b^a) is modular preserving:

$$(z_b^a) \in SU(N^2 - 1).$$

This leads us to study the covariant theory for complex rotations of gauge fields corresponding to the Euclid metric $G_{ab} = \delta_{ab}$ as follows.

Let τ_a be the generators of $SU(N)$ given by (3.25). Then we take the unitary transformation

$$(3.27) \quad \tilde{\tau}_b = z_{ba} \tau_a, \quad (z_{ba}) \in SU(N^2 - 1).$$

For the orthogonal transformation, the $SU(N)$ -tensors λ_{ab}^c have no distinction between contra-variant and covariant indices, i.e.

$$\lambda_{ab}^c = \lambda_{abc}.$$

Therefore the metric tensors G_{ab} can be written as

$$G_{ab} = \lambda_{acd} \lambda_{dcb}^*,$$

where λ^* is the complex conjugate of λ , and λ_{acd} transform as

$$(3.28) \quad \tilde{\lambda}_{abc} = z_{ad} z_{bf} z_{cg} \lambda_{dfg}.$$

Thus \tilde{G}_{ab} is as follows

$$(\tilde{G}_{ab}) = (\tilde{\lambda}_{acd} \tilde{\lambda}_{dcb}^*) = (z_{ab})(G_{ab})(z_{ab})^\dagger = (\delta_{ab}),$$

thanks to $G_{ab} = \delta_{ab}$. Hence, under the unitary transformation (3.27), the $SU(N)$ metric (G_{ab}) is invariant. The corresponding unitary transformations of A_μ^a and θ^a are given by

$$(3.29) \quad \tilde{A}_\mu^a = z_{ab} A_\mu^b, \quad \tilde{\theta}^a = z_{ab} \theta^b.$$

Thus, for the unitary transformations (3.27)-(3.29), the invariant gauge action (3.20) becomes

$$(3.30) \quad \begin{aligned} L_G &= \int [\mathcal{L}_G + \mathcal{L}_F] dx, \\ \mathcal{L}_G &= g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}^{a\dagger} F_{\alpha\beta}^a, \\ \mathcal{L}_F &= \bar{\Psi} [i\gamma^\mu (\partial_\mu - ig A_\mu^{a\dagger} \tau_a) - m] \Psi, \end{aligned}$$

where $F_{\mu\nu}^a$ is as in (3.9) with $\lambda^{abc} = \lambda_{abc}$.

3.5. Remarks. In summary, we have shown the following theorem, providing the needed mathematical foundation for PRI.

Theorem 3.1. *For $SU(N)$ ($N \geq 2$), the following assertions hold true:*

- (1) *For each representation of $SU(N)$ with generators $\{\tau_a\}$, the $SU(N)$ -tensor (G_{ab}) is symmetric and positive definite. Consequently, (G_{ab}) can be defined on $SU(N)$ as a Riemannian metric, and the action (3.20) is a unique form which obeys the Lorentz invariance, the gauge invariance of the transformation (3.13), and the PRI.*
- (2) *If $\{\tau_a\}$ is taken as in (3.25), the metric (G_{ab}) is Euclidian, and is invariant under the unitary transformation (3.27). Moreover, the corresponding unitary invariant action takes the form (3.30).*

PRI provides a strong restriction on gauge field theories, and we address now some direct consequences of such restrictions.

We know that the standard model for the electroweak and strong interactions is a $U(1) \times SU(2) \times SU(3)$ gauge theory combined with the Higgs mechanism. A remarkable character for the Higgs mechanism is that the gauge fields with different symmetry groups are combined linearly into terms in the corresponding gauge field equations. For example, in the Weinberg-Salam electroweak gauge equations with $U(1) \times SU(2)$ symmetry breaking, there are such linearly combined terms as

$$\begin{aligned}
 Z_\mu &= \cos \theta_w W_\mu^3 + \sin \theta_w B_\mu, \\
 A_\mu &= -\sin \theta_w W_\mu^3 + \cos \theta_w B_\mu, \\
 W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2),
 \end{aligned}
 \tag{3.31}$$

where W_μ^a ($1 \leq a \leq 3$) are $SU(2)$ gauge fields, and B_μ is a $U(1)$ gauge field. It is clear that these terms (3.31) are not covariant under the general unitary transformation as given in (3.27). Hence the classical Higgs mechanism violates the PRI. As the standard model is based on the classical Higgs mechanism, it violates PRI and can only be considered an approximate model describing interactions in nature.

The grand unification theory (GUT) puts $U(1) \otimes SU(2) \otimes SU(3)$ into $SU(5)$ or $O(10)$, whose gauge fields correspond to some specialized representations of $SU(5)$ or $O(10)$ generators. Since similar Higgs are crucial for GUT, it is clear that GUT violates PRI as well.

As far as we know, it appears that the only unified field model, which obeys PRI, is the unified field theory based on PID presented in this article, from which we can derive not only the same physical conclusions as those from the standard model, but also many new results and predictions, leading to the solution of a number of longstanding open questions in particle physics.

4. UNIFIED FIELD MODEL BASED ON PID AND PRI.

4.1. Unified field equations obeying PRI. In [12], we derived a set of unified field equations coupling four interactions based on PID. In view of PRI, we now refine this model, ensuring that these equations are covariant under the $U(1) \otimes SU(2) \otimes SU(3)$ generator transformations.

The action functional is the natural combination of the Einstein-Hilbert functional, the QED action, the weak interaction action, and the standard QCD action:

$$(4.1) \quad L = \int [\mathcal{L}_{EH} + \mathcal{L}_{QED} + \mathcal{L}_W + \mathcal{L}_{QCD}] \sqrt{-g} dx,$$

where

$$(4.2) \quad \begin{aligned} \mathcal{L}_{EH} &= R + \frac{8\pi G}{C^4} S, \\ \mathcal{L}_{QED} &= -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \bar{\psi}(i\gamma^\mu \tilde{D}_\mu - m)\psi, \\ \mathcal{L}_W &= -\frac{1}{4} G_{ab}^w g^{\mu\alpha} g^{\nu\beta} W_{\mu\nu}^a W_{\alpha\beta}^b + \bar{L}(i\gamma^\mu \tilde{D}_\mu - m^l)L, \\ \mathcal{L}_{QCD} &= -\frac{1}{4} G_{ab}^s g^{\mu\alpha} g^{\nu\beta} S_{\mu\nu}^a S_{\alpha\beta}^b + \bar{q}(i\gamma^\mu \tilde{D}_\mu - m^q)q. \end{aligned}$$

Here R is the scalar curvature of the space-time Riemannian manifold $(M, g_{\mu\nu})$ with Minkowski signature, S is the energy-momentum density, G_{ab}^w and G_{ab}^s are the metrics of $SU(2)$ and $SU(3)$ as defined by (3.19), ψ are the wave functions of charged fermions, $L = (L_1, L_2)^T$ are the wave functions of left-hand lepton and quark pairs (each has 3 generations), $q = (q_1, q_2, q_3)^T$ are the flavored quarks, and

$$(4.3) \quad \begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu, \\ W_{\mu\nu}^a &= \nabla_\mu W_\nu^a - \nabla_\nu W_\mu^a + g_w \lambda_{bc}^a W_\mu^b W_\nu^c, \\ S_{\mu\nu}^a &= \nabla_\mu S_\nu^a - \nabla_\nu S_\mu^a + g_s \Lambda_{bc}^a S_\mu^b S_\nu^c. \end{aligned}$$

Here A_μ is the electromagnetic potential, W_μ^a ($1 \leq a \leq 3$) are the $SU(2)$ gauge fields for the weak interaction, S_μ^a ($1 \leq a \leq 8$) are the $SU(3)$ gauge fields for QCD, ∇_μ is the Levi-Civita covariant derivative, and

$$(4.4) \quad \begin{aligned} \tilde{D}_\mu L &= (\tilde{\nabla}_\mu + ieA_\mu + ig_w W_\mu^a \sigma_a)L, \\ \tilde{D}_\mu \psi &= (\tilde{\nabla}_\mu + ieA_\mu)\psi, \\ \tilde{D}_\mu q &= (\tilde{\nabla}_\mu + ig_s S_\mu^b \tau_b)q, \end{aligned}$$

where $\tilde{\nabla}_\mu$ is the Lorentz Vierbein covariant derivative [6], σ_a ($1 \leq a \leq 3$) are the generators of $SU(2)$, and τ_b ($1 \leq b \leq 8$) are the generators of $SU(3)$.

We can show that, for a gauge field A_μ and an antisymmetric tensor field $F_{\mu\nu}$, we have

$$(4.5) \quad \begin{aligned} \nabla_\mu A_\nu - \nabla_\nu A_\mu &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ \nabla_\mu F_{\nu\lambda} - \nabla_\nu F_{\mu\lambda} + \nabla_\lambda F_{\mu\nu} &= \partial_\mu F_{\nu\lambda} - \partial_\nu F_{\mu\lambda} + \partial_\lambda F_{\mu\nu}. \end{aligned}$$

It is easy to see that the action (4.1) obeys the principle of general relativity, and is invariant under Lorentz (Vierbein) transformation, and the $U(1) \times SU(2) \times SU(3)$

gauge transformation:

$$\begin{aligned}
(4.6) \quad & A_\mu \rightarrow A_\mu - \frac{1}{e} \tilde{\nabla}_\mu \theta, \\
& W_\mu^a \rightarrow W_\mu^a - \frac{1}{g_w} \tilde{\nabla}_\mu \theta^a + \lambda_{bc}^a \theta^b w_\mu^c, \\
& S_\mu^a \rightarrow S_\mu^a - \frac{1}{g_s} \tilde{\nabla}_\mu \phi^a + \Lambda_{bc}^a \phi^b S_\mu^c, \\
& \psi \rightarrow e^{i\theta} \psi, \\
& L \rightarrow e^{i\theta^a \sigma_a} L, \\
& q \rightarrow e^{i\phi^a \tau_a} q, \\
& m^l \rightarrow e^{i\phi^a \tau_a} m^l e^{-i\phi^a \tau_a}.
\end{aligned}$$

Also, the action (4.1) is invariant under the transformations of $SU(2)$ and $SU(3)$ generators σ_a and τ_a :

$$\begin{aligned}
(4.7) \quad & \sigma_a \rightarrow x_a^b \sigma_b, \quad (x_a^b) \in GL(\mathbb{C}^3), \\
& \tau_a \rightarrow y_a^b \tau_b, \quad (y_a^b) \in GL(\mathbb{C}^8),
\end{aligned}$$

where $GL(\mathbb{C}^n)$ is the general linear group of all $n \times n$ non-degenerate complex matrices.

We are now in position to establish unified field equations with PRI covariance. By PID and PRI, the unified model should be taken by the variation of the action (4.1) under the div_A -free constraint

$$(\delta L, X) = 0 \quad \text{for any } X \text{ with } \text{div}_A X = 0.$$

Here it is required that the gradient operators ∇_A corresponding to div_A are co-variant under transformation (4.7). Therefore we have

$$\begin{aligned}
(4.8) \quad & D_\mu^G = \nabla_\mu - \alpha^0 A_\mu - \alpha_b^1 W_\mu^b - \alpha_k^2 S_\mu^k, \\
& D_\mu^E = \nabla_\mu - \beta^0 A_\mu - \beta_b^1 W_\mu^b - \beta_k^2 S_\mu^k, \\
& D_\mu^w = \nabla_\mu - \gamma^0 A_\mu - \gamma_b^1 W_\mu^b - \gamma_k^2 S_\mu^k + \frac{m_w^2}{4} x_\mu, \\
& D_\mu^s = \nabla_\mu - \delta^0 A_\mu - \delta_b^1 W_\mu^b - \delta_k^2 S_\mu^k + \frac{m_s^2}{4} x_\mu,
\end{aligned}$$

where

$$\begin{aligned}
(4.9) \quad & m_w, m_s, \alpha^0, \beta^0, \gamma^0, \delta^0 \quad \text{are scalar parameters,} \\
& \alpha_a^1, \beta_a^1, \gamma_a^1, \delta_a^1 \quad \text{are the } SU(2) \text{ order-1 tensors,} \\
& \alpha_k^2, \beta_k^2, \gamma_k^2, \delta_k^2 \quad \text{are the } SU(3) \text{ order-1 tensors.}
\end{aligned}$$

Thus, (2.7)-(2.8) can be expressed as

$$(4.10) \quad \begin{aligned} \frac{\delta L}{\delta g_{\mu\nu}} &= D_\mu^G \Phi_\nu^G, \\ \frac{\delta L}{\delta A_\mu} &= D_\mu^E \phi^E, \\ \frac{\delta L}{\delta W_\mu^a} &= D_\mu^w \phi_w^a, \\ \frac{\delta L}{\delta S_\mu^k} &= D_\mu^s \phi_s^k, \end{aligned}$$

where Φ_ν^G is a vector field, and $\phi^E, \phi_w^a, \phi_s^k$ are scalar fields.

Then, the unified model with PRI covariance is derived from (4.1)-(4.5), (4.10) and (2.9) as

$$(4.11) \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{8\pi G}{c^4}T_{\mu\nu} = D_\mu^G \Phi_\nu^G,$$

$$(4.12) \quad \partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) - eJ_\nu = D_\nu^E \phi^E,$$

$$(4.13) \quad G_{ab}^w \left[\partial^\mu W_{\mu\nu}^b - \frac{g_w}{2} \lambda_{cd}^b g^{\alpha\beta} W_{\alpha\nu}^c W_\beta^d \right] - g_w J_{\nu a} = D_\nu^w \phi_a^w,$$

$$(4.14) \quad G_{kj}^s \left[\partial^\mu S_{\mu\nu}^j - \frac{g_s}{2} \Lambda_{cd}^j g^{\alpha\beta} S_{\alpha\nu}^c S_\beta^d \right] - g_s Q_{\nu k} = D_\nu^s \phi_k^s,$$

$$(4.15) \quad (i\gamma^\mu \tilde{D}_\mu - m^l)L = 0,$$

$$(4.16) \quad (i\gamma^\mu \tilde{D}_\mu - m)\psi = 0,$$

$$(4.17) \quad (i\gamma^\mu \tilde{D}_\mu - m^q)q = 0,$$

where $D_\mu^G, D_\mu^E, D_\mu^w, D_\mu^s$ are given by (4.8), and

$$(4.18) \quad J_\nu = \bar{\psi}\gamma_\nu\psi, \quad J_{\nu a} = \bar{L}\gamma_\nu\sigma_a L, \quad Q_{\nu k} = \bar{q}\gamma_\nu\tau_k q,$$

$$(4.19) \quad \begin{aligned} T_{\mu\nu} &= \frac{\delta S}{\delta g_{\mu\nu}} + \frac{c^4}{16\pi G} g^{\alpha\beta} (G_{ab}^w W_{\alpha\mu}^a W_{\beta\nu}^b + G_{ab}^s S_{\alpha\mu}^a S_{\beta\nu}^b + F_{\alpha\mu} F_{\beta\nu}) \\ &\quad - \frac{c^4}{16\pi G} g_{\mu\nu} (\mathcal{L}_{QED} + \mathcal{L}_W + \mathcal{L}_{QCD}). \end{aligned}$$

4.2. Coupling parameters. The equations (4.11)-(4.17) are in general form where the $SU(2)$ and $SU(3)$ generators σ_a and τ_a are taken arbitrarily. If we take σ_a ($1 \leq a \leq 3$) as the Pauli matrices (3.21), and $\tau_a = \lambda_a$ ($1 \leq a \leq 8$) as the Gell-Mann matrices (3.22), then both metrics

$$(4.20) \quad G_{ab}^w = \delta_{ab} \ (1 \leq a, b \leq 3), \quad G_{ab}^s = \delta_{ab} \ (1 \leq a, b \leq 8)$$

are the Euclidian, and there is no need to distinguish the $SU(N)$ covariant tensors and contra-variant tensors.

Hence in general we usually take the Pauli matrices σ_a and the Gell-Mann matrices λ_k as the $SU(2)$ and $SU(3)$ generators. For convenience we introduce dimensions of related physical quantities. Let E represent energy, L be the length and t

be the time. Then we have

$$\begin{aligned} (A_\mu, W_\mu^a, S_\mu^k) &: \sqrt{E/L}, & (e, g_w, g_s) &: \sqrt{EL}, \\ (J_\mu, J_{\mu a}, Q_{\mu k}) &: 1/L^3, & (\phi^E, \phi_w^a, \phi_s^k) &: \frac{\sqrt{E}}{\sqrt{LL}}, \\ \hbar &: Et, & c &: L/t, & mc/\hbar &: 1/L \quad (m \text{ the mass}). \end{aligned}$$

Thus the parameters in (4.9) can be rewritten as

$$\begin{aligned} (m_w, m_s) &= \left(\frac{m_H c}{\hbar}, \frac{m_\pi c}{\hbar} \right), \\ (\alpha^0, \beta^0, \gamma^0, \delta^0) &= \frac{e}{\hbar c} (\alpha^E, \beta^E, \gamma^E, \delta^E), \\ (\alpha_a^1, \beta_a^1, \gamma_a^1, \delta_a^1) &= \frac{g_w}{\hbar c} (\alpha_a^w, \beta_a^w, \gamma_a^w, \delta_a^w), \\ (\alpha_k^2, \beta_k^2, \gamma_k^2, \delta_k^2) &= \frac{g_s}{\hbar c} (\alpha_k^s, \beta_k^s, \gamma_k^s, \delta_k^s), \end{aligned} \quad (4.21)$$

where m_H and m_π represent the masses of ϕ^w and ϕ^s , and all the parameters $(\alpha, \beta, \gamma, \delta)$ on the right hand side with different super and sub indices are dimensionless constants.

It is worth mentioning that the to-be-determined coupling parameters lead to the discovery of PRI. Perhaps there are still some undiscovered physical principles or rules which can reduce the number of parameters in (4.21).

Then the unified field equations (4.11)-(4.17) can be simplified in the form

$$(4.22) \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} + \left[\nabla_\mu - \frac{e\alpha^E}{\hbar c} A_\mu - \frac{g_w \alpha_b^w}{\hbar c} W_\mu^b - \frac{g_s \alpha_k^s}{\hbar c} S_\mu^k \right] \Phi_\nu,$$

$$(4.23) \quad \partial^\nu F_{\nu\mu} = eJ_\mu + \left[\nabla_\mu - \frac{e\beta^E}{\hbar c} A_\mu - \frac{g_w \beta_b^w}{\hbar c} W_\mu^b - \frac{g_s \beta_k^s}{\hbar c} S_\mu^k \right] \phi^E,$$

$$\begin{aligned} (4.24) \quad \partial^\nu W_{\nu\mu}^a - \frac{g_w}{2\hbar c} \varepsilon^{abc} g^{\alpha\beta} W_{\alpha\mu}^b W_\beta^c - g_w J_\mu^a \\ = \left[\nabla_\mu - \frac{e\gamma^E}{\hbar c} A_\mu - \frac{g_w \gamma_b^w}{\hbar c} W_\mu^b - \frac{g_s \gamma_k^s}{\hbar c} S_\mu^k + \frac{1}{4} \left(\frac{m_H c}{\hbar} \right)^2 x_\mu \right] \phi_w^a, \end{aligned}$$

$$\begin{aligned} (4.25) \quad \partial^\nu S_{\nu\mu}^k - \frac{g_s}{2\hbar c} f^{kij} g^{\alpha\beta} S_{\alpha\mu}^i S_\beta^j - g_s Q_\mu^k \\ = \left[\nabla_\mu - \frac{e\delta^E}{\hbar c} A_\mu - \frac{g_w \delta_b^w}{\hbar c} W_\mu^b - \frac{g_s \delta_l^s}{\hbar c} S_\mu^l + \frac{1}{4} \left(\frac{m_\pi c}{\hbar} \right)^2 x_\mu \right] \phi_s^k, \end{aligned}$$

$$(4.26) \quad (i\gamma^\mu D_\mu - \tilde{m})\Psi = 0,$$

where $\Psi = (\psi, L, q)$, $F_{\mu\nu}$ is as in (4.3), and

$$\begin{aligned} (4.27) \quad W_{\nu\mu}^a &= \partial_\nu W_\mu^a - \partial_\mu W_\nu^a + \frac{g_w}{\hbar c} \varepsilon^{abc} W_\nu^b W_\mu^c, \\ S_{\nu\mu}^k &= \partial_\nu S_\mu^k - \partial_\mu S_\nu^k + \frac{g_s}{\hbar c} f^{kij} S_\nu^i S_\mu^j. \end{aligned}$$

Equations (4.22)-(4.26) need to be supplemented with coupled gauge equations to fix the gauge to compensate the symmetry-breaking and the induced adjoint fields $(\phi^E, \phi_w^a, \phi_s^k)$. In different physical situations, the coupled gauge equations may be different. However, they usually take the following form:

$$(4.28) \quad \partial^\mu A_\mu = 0, \quad \partial^\mu W_\mu^a = \text{constant}, \quad \partial^\mu S_\mu^k = \text{constant}.$$

From the physical point of view, the coefficients $\alpha^E, \beta^E, \gamma^E, \delta^E$ should be the same:

$$(4.29) \quad \alpha^E = \beta^E = \gamma^E = \delta^E,$$

depending on the energy density. For the $SU(2)$ and $SU(3)$ vector constants, it is natural to take

$$(4.30) \quad \begin{aligned} \alpha_a^w &= \beta_a^w = \gamma_a^w = \delta_a^w & \text{for } 1 \leq a \leq 3, \\ \alpha_k^s &= \beta_k^s = \gamma_k^s = \delta_k^s & \text{for } 1 \leq k \leq 8. \end{aligned}$$

Therefore, by (4.29) and (4.30), the to-be-determined parameters reduce to the following $SU(2)$ and $SU(3)$ vectors:

$$(4.31) \quad \{\alpha_a^w\} = (\alpha_1^w, \alpha_2^w, \alpha_3^w) \quad \text{and} \quad \{\alpha_k^s\} = (\alpha_1^s, \dots, \alpha_8^s),$$

consisting of 11 to-be-determined constants. In particular, in two accompanying papers on weak and strong interactions, we find that each component of $\{\alpha_a^w\}$ and $\{\alpha_k^s\}$ represents the portion distributed to the gauge potential W_μ^a and S_μ^k by the weak and strong charges g_w and g_s . Consequently, we have

$$(4.32) \quad |\{\alpha_a^w\}| = \sqrt{\alpha_a^w \alpha_a^w} = \alpha^w, \quad |\{\alpha_k^s\}| = \sqrt{\alpha_k^s \alpha_k^s} = \alpha^s.$$

The strength scalar parameters α^E, α^w and α^s depend on the energy density, and for decoupled interaction, they all are given by

$$(4.33) \quad \alpha^E = \alpha^w = \alpha^s = 1.$$

Hence finally, we derive the following

$$(4.34) \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{8\pi G}{c^4}T_{\mu\nu} = \left[\nabla_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_a^w}{\hbar c}W_\mu^a - \frac{g_s\alpha_k^s}{\hbar c}S_\mu^k \right] \Phi_\nu,$$

$$(4.35) \quad \partial^\nu F_{\nu\mu} = \left[\nabla_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_a^w}{\hbar c}W_\mu^a - \frac{g_s\alpha_k^s}{\hbar c}S_\mu^k \right] \phi^E + eJ_\mu,$$

$$(4.36) \quad \begin{aligned} \partial^\nu W_{\nu\mu}^a - \frac{g_w}{2\hbar c}\varepsilon_{bc}^a g^{\alpha\beta} W_{\alpha\mu}^b W_\beta^c - g_w J_\mu^a \\ = \left[\nabla_\mu + \frac{1}{4} \left(\frac{m_H c}{\hbar} \right)^2 x_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_b^w}{\hbar c}W_\mu^b - \frac{g_s\alpha_k^s}{\hbar c}S_\mu^k \right] \phi_w^a, \end{aligned}$$

$$(4.37) \quad \begin{aligned} \partial^\nu S_{\nu\mu}^k - \frac{g_s}{2\hbar c}f^{kij} g^{\alpha\beta} S_{\alpha\mu}^i S_\beta^j - g_s Q_\mu^k \\ = \left[\nabla_\mu + \frac{1}{4} \left(\frac{m_\pi c}{\hbar} \right)^2 x_\mu - \frac{e\alpha^E}{\hbar c}A_\mu - \frac{g_w\alpha_a^w}{\hbar c}W_\mu^a - \frac{g_s\alpha_j^s}{\hbar c}S_\mu^j \right] \phi_s^k, \end{aligned}$$

$$(4.38) \quad (i\gamma^\mu D_\mu - \tilde{m})\Psi = 0.$$

5. DUALITY AND DECOUPLING OF INTERACTING FIELDS

5.1. Duality. In [12], we have obtained a natural duality between the interacting fields (g, A, W^a, S^k) and their adjoint fields $(\Phi_\mu, \phi^E, \phi_a^w, \phi_k^s)$ as follows

$$(5.1) \quad \begin{aligned} \{g_{\mu\nu}\} &\longleftrightarrow \Phi_\mu, \\ A_\mu &\longleftrightarrow \phi^E, \\ W_\mu^a &\longleftrightarrow \phi_w^a \quad \forall 1 \leq a \leq 3, \\ S_\mu^k &\longleftrightarrow \phi_s^k \quad \forall 1 \leq k \leq 8. \end{aligned}$$

However, due to the discovery of PRI symmetry, the $SU(2)$ gauge fields W_μ^a ($1 \leq a \leq 3$) and the $SU(3)$ gauge fields S_μ^b ($1 \leq b \leq 8$) are symmetric in their indices $a = 1, 2, 3$ and $b = 1, \dots, 8$ respectively. Therefore the corresponding relation (5.1) is changed into the following dual relation

$$(5.2) \quad \begin{aligned} \{g_{\mu\nu}\} &\longleftrightarrow \Phi_\mu, \\ A_\mu &\longleftrightarrow \phi^E, \\ \{W_\mu^a\} &\longleftrightarrow \{\phi_w^a\}, \\ \{S_\mu^k\} &\longleftrightarrow \{\phi_s^k\}. \end{aligned}$$

In comparison to the duality (5.1) discovered in [12], the new viewpoint here is that the three fields W_μ^a ($a = 1, 2, 3$) and the eight fields S_μ^k ($1 \leq k \leq 8$) are regarded as two gauge group tensors corresponding to $SU(2)$ and $SU(3)$ tensor fields: $\{\phi_w^a\}$ and $\{\phi_s^k\}$ respectively. This change is caused by the PRI symmetry, leading to the PRI covariant field equations (4.11)-(4.17).

In [10, 12], we have discussed the interaction between the gravitational field $\{g_{ij}\}$ and its adjoint field $\{\Phi_\mu\}$, leading to a unified theory for dark energy and dark matter. Hereafter we focus on the electromagnetic pair A_μ and ϕ^E , the weak interaction pair $\{w_\mu^a\}$ and $\{\phi_w^a\}$, and the strong interaction pair $\{S_\mu^k\}$ and $\{\phi_s^k\}$.

An important case is that

$$(5.3) \quad \phi_w^a = \eta^a \phi_w,$$

$$(5.4) \quad \phi_s^k = \zeta^k \phi_s,$$

where η_a and ζ_k are constant representation vectors.

5.2. Modified QED model. For the electromagnetic interaction only, the decoupled QED field equations from (4.23), (4.28) and (4.29) are given by

$$(5.5) \quad \frac{1}{c^2} \frac{\partial^2 A_\mu}{\partial t^2} - \nabla^2 A_\mu = eJ_\mu + \left[\partial_\mu - \frac{\alpha^E e}{\hbar c} A_\mu \right] \phi^E,$$

$$(5.6) \quad i\gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\psi = 0,$$

$$(5.7) \quad \partial^\mu A_\mu = 0,$$

where $\alpha^E = \pm 1$, $J_\mu = \bar{\psi} \gamma_\mu \psi$ is the current satisfying

$$\partial^\mu J_\mu = 0.$$

Equations (5.5)-(5.7) are the modified QED model, which can also be written as

$$(5.8) \quad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) A_\mu + \frac{\alpha^E e}{\hbar c} \phi^E A_\mu = eJ_\mu + \partial_\mu \phi^E,$$

$$(5.9) \quad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi^E - \frac{\alpha^E e}{\hbar c} A_\mu \cdot \partial^\mu \phi^E = 0,$$

$$(5.10) \quad i\gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\psi = 0,$$

$$(5.11) \quad \partial^\mu A_\mu = 0.$$

If we take the form

$$(5.12) \quad H = \text{curl} \vec{A}, \quad E = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla \varphi,$$

where $A_\mu = (\varphi, \vec{A})$, $\vec{A} = (A_1, A_2, A_3)$, then the equations (4.12) and (5.11) are a modified version of the Maxwell equations expressed as

$$(5.13) \quad \frac{1}{c} \frac{\partial H}{\partial t} = -\text{curl} E,$$

$$(5.14) \quad H = \text{curl} \vec{A},$$

$$(5.15) \quad \frac{1}{c} \frac{\partial E}{\partial t} = \text{curl} H + \vec{J} + \nabla \phi^E - \frac{\alpha^E e}{\hbar c} \phi^E \vec{A},$$

$$(5.16) \quad \text{div} E = \rho + \frac{1}{c} \frac{\partial \phi^E}{\partial t} - \frac{\alpha^E e}{\hbar c} \phi^E \varphi,$$

$$(5.17) \quad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi^E - \frac{\alpha^E e}{\hbar c} \left(\frac{1}{c} \varphi \frac{\partial \phi^E}{\partial t} - \vec{A} \cdot \nabla \phi^E \right) = 0,$$

where $\vec{J} = (J_1, J_2, J_3)$ is the electric current density and ρ is the electric charge density. Equations (5.13)-(5.17) need to be supplemented with a coupled equation to fix the gauge compensating the symmetry breaking and the induced adjoint field ϕ^E .

5.3. Weak interactions. We derive now the field equations for weak interaction using an $SU(2)$ gauge theory based on PID and PRI. The action functional is

$$(5.18) \quad L_W = \int \mathcal{L}_W dx,$$

where

$$(5.19) \quad \mathcal{L}_W = -\frac{1}{4} G_{ab}^w g^{\mu\alpha} g^{\nu\beta} W_{\mu\nu}^a W_{\alpha\beta}^b + \bar{L}(i\gamma^\mu D_\mu - m^l)L.$$

Here G_{ab}^w is the metric defined by (3.19), $L = (L_1, L_2)^T$ are the wave functions of left-hand lepton and quark pairs (each has 3 generations), and

$$(5.20) \quad W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \frac{g_w}{\hbar c} \lambda_{bc}^a W_\mu^b W_\nu^c.$$

Here W_μ^a ($1 \leq a \leq 3$) are the $SU(2)$ gauge fields for the weak interaction, and

$$(5.21) \quad D_\mu L = (\partial_\mu + ig_w W_\mu^a \tau_a)L,$$

where τ_a ($1 \leq a \leq 3$) are the generators of $SU(2)$.

Using PID, the weak interaction field equations are given by

$$(5.22) \quad G_{ab}^w \left[\partial^\mu W_{\mu\nu}^b - \frac{g_w}{2\hbar c} \lambda_{cd}^b g^{\alpha\beta} W_{\alpha\nu}^c W_\beta^d \right] - g_w J_{\nu a} \\ = \left[\partial_\nu - \frac{g_w}{\hbar c} \alpha_b^w W_\nu^b + \frac{k_0^2}{4} x_\nu \right] \phi_a^w,$$

$$(5.23) \quad (i\gamma^\mu \tilde{D}_\mu - m^l)L = 0,$$

where $J_{\nu a} = \bar{L}\gamma_\nu \tau_a L$, k_0 is a constant, $k_0^2 x_\mu/4$ is a mass potential, g_w is the weak charge, and α_a^w is the $SU(2)$ vector representing the portions distributed to the gauge potentials by the weak charge.

The above field equations readily lead to a natural duality:

$$(5.24) \quad \{W_\mu^a\} \longleftrightarrow \{\phi_w^a\}.$$

The left side of this duality induces the intermediate vector bosons W^\pm and Z , and the right had side gives rise to three Higgs bosons: one neutral and two charged.

We can separate the field equations for ϕ_w^a by taking divergence on both sides of (5.22). Take the Pauli matrices as the generators of $SU(2)$, and notice that

$$\partial^\mu \partial^\nu W_{\mu\nu}^a = 0 \quad \forall 1 \leq a \leq 3.$$

Then we derive that

$$(5.25) \quad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi_w^a + \left(\frac{m_H c}{\hbar} \right)^2 \phi_w^a + \frac{1}{4} \left(\frac{m_H c}{\hbar} \right)^2 x_\mu \partial^\mu \phi_w^a \\ - \frac{g_w}{\hbar c} \alpha_b^w \partial^\mu (W_\mu^b \phi_w^a) = g_w \partial^\mu J_\mu^a - \frac{g_w}{2\hbar c} g^{\alpha\alpha} \varepsilon^{abc} \partial^\mu (W_{\alpha\mu}^b W_\alpha^c).$$

Another possible duality is the degenerate case where the three scalar fields ϕ_a^w are a constant vector ζ_a times a single scalar field ϕ^w : $\phi_a^w = \frac{g_w}{\sqrt{\hbar c}} \zeta_a \phi^w$. In this case, the duality reduces to

$$(5.26) \quad \{W_\mu^a\} \longleftrightarrow \{\phi^w\}.$$

Again, the left side of this duality induces the intermediate vector bosons W^\pm and Z . However, the right had side gives rise to one neutral Higgs boson.

For the duality (5.26), if we take the Pauli matrices as the generators of $SU(2)$, then (5.22) can be rewritten as

$$(5.27) \quad \partial^\mu W_{\mu\nu}^a - \frac{g_w}{2\hbar c} \varepsilon^{abc} g^{\alpha\beta} W_{\alpha\nu}^b W_\beta^c - g_w J_\nu^a = \frac{g_w}{\sqrt{\hbar c}} \zeta^a \left[\partial_\mu - \frac{g_w}{\hbar c} \alpha_b^w W_\nu^b + \frac{k_0^2}{4} x_\nu \right] \phi^w.$$

5.4. Strong interactions. The decoupled model for strong interaction is given by:

$$(5.28) \quad G_{kj}^s \left[\partial^\mu S_{\mu\nu}^j - \frac{g_s}{2} \Lambda_{cd}^j g^{\alpha\beta} S_{\alpha\nu}^c S_\beta^d \right] - g_s Q_{\nu k} \\ = \left[\partial_\nu + \frac{1}{4} \left(\frac{m_\pi c}{\hbar} \right)^2 x_\nu - \frac{g_s}{\hbar c} \alpha_j^s S_\nu^j \right] \phi_k^s,$$

$$(5.29) \quad i\gamma^\mu (\hbar c \partial_\mu + i g_s S_\mu^k \lambda_k) q - m_q c^2 q = 0,$$

where $\{\alpha_k^s\} = (\alpha_1^s, \dots, \alpha_8^s)$ is the $SU(3)$ constant vector, and

$$(5.30) \quad S_{\mu\nu}^j = \partial_\mu S_\nu^j - \partial_\nu S_\mu^j + \frac{g_s}{\hbar c} \Lambda_{kl}^j S_\mu^k S_\nu^l.$$

For the strong interactions duality

$$\{S_\mu^k\} \longleftrightarrow \{\phi_s^k\},$$

if we take $\tau_a = \lambda_a$ ($1 \leq a \leq 8$) as the Gell-Mann matrices (3.22), the field equations derived from (4.25) and (4.26) are given by

$$(5.31) \quad \partial^\nu S_{\nu\mu}^k - \frac{g_s}{2\hbar c} f^{kij} g^{\alpha\beta} S_{\alpha\mu}^i S_\beta^j - g_s Q_\mu^k = \left[\partial_\mu + \frac{1}{4} \left(\frac{m_\pi c}{\hbar} \right)^2 x_\mu - \frac{g_s \alpha_j^s}{\hbar c} S_\mu^j \right] \phi_s^k,$$

$$(5.32) \quad (i\gamma^\mu D_\mu - m) q = 0,$$

where m_π is the mass of the Yukawa meson, f^{bcd} are the structural constants as in (3.23),

$$(5.33) \quad Q_\mu^k = \bar{q} \gamma_\mu \lambda^k q \quad (1 \leq k \leq 8)$$

are quark currents, m is the quark mass, and

$$D_\mu = \partial_\mu + i g_s S_\mu^k \lambda^k.$$

Taking divergence on both sides of (5.31), we derive the equation for the adjoint fields ϕ_s^k as follows

$$(5.34) \quad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi_s^k + \left(\frac{m_\pi c}{\hbar} \right)^2 \phi_s^k + \frac{1}{4} \left(\frac{m_\pi c}{\hbar} \right)^2 x_\mu \partial^\mu \phi_s^k - \frac{g_s \alpha_j^s}{\hbar c} \partial^\mu (S_\mu^j \phi_s^k) = g_s \partial^\mu Q_\mu^k - \frac{g_s}{2\hbar c} f^{kij} g^{\alpha\alpha} \partial^\mu (S_{\alpha\mu}^i S_\alpha^j).$$

Equations (5.31), (5.32) and (5.34) are the duality model for strong interacting fields.

If we consider the duality (5.4), the field equation (5.31) is expressed as

$$(5.35) \quad \partial^\nu S_{\nu\mu}^k - \frac{g_s}{2\hbar c} f^{kij} g^{\alpha\alpha} S_{\alpha\mu}^i S_\alpha^j - g_s Q_\mu^k = \frac{g_s}{\sqrt{\hbar c}} \zeta^k \left[\partial_\mu + \frac{1}{4} \left(\frac{m_\pi c}{\hbar} \right)^2 x_\mu \right] \phi^s.$$

Here we ignored the coupling to S_μ^k due to the fact that gluons are massless. Taking divergence on both sides of (5.35) and making the contraction with $\{\zeta^k\}$ we have

$$(5.36) \quad \partial^\mu \partial_\mu \phi^s + \left(\frac{m_\pi c}{\hbar} \right)^2 \phi^s = -\frac{\sqrt{\hbar c}}{|\zeta|^2} \zeta^k \partial^\mu Q_\mu^k - \frac{1}{4} \left(\frac{m_\pi c}{\hbar} \right)^2 x_\mu \partial^\mu \phi^s - \frac{1}{2\sqrt{\hbar c}} \frac{\zeta^k}{|\zeta|^2} f^{kij} g^{\alpha\beta} \partial^\mu (S_{\alpha\mu}^i S_\beta^j).$$

6. QUARK POTENTIALS

6.1. Strong acting forces. We know that the electromagnetism is caused by the electric charge e , and the coupling constant of the $U(1)$ gauge field. In particular, the electromagnetic potential $A_\mu = (A_0, A_1, A_2, A_3)$ can be interpreted as:

$$(6.1) \quad \begin{aligned} A_0 &= \Phi && \text{the electric potential,} \\ \vec{A} &= (A_1, A_2, A_3) && \text{the magnetic potential,} \end{aligned}$$

and

$$(6.2) \quad \begin{aligned} F_e &= -e \nabla \Phi && \text{the force acting on particles with charge } e, \\ F_m &= \frac{1}{c} e \vec{v} \times \text{curl} \vec{A} && \text{the Lorentz force acting on } e. \end{aligned}$$

In the same spirit as the electromagnetism, the strong interaction is modeled by an $SU(3)$ gauge theory. The gauge potential S_μ consists of eight constituents of vector fields:

$$S_\mu = \{S_\mu^k \mid 1 \leq k \leq 8\}, \quad S_\mu^k = (S_0^k, S_1^k, S_2^k, S_3^k),$$

and the k -th constituent S_μ^k corresponds to the k -th gluon. The coupling constant g_s of $SU(3)$ gauge fields plays a similar role as the electric charge e , and is called the strong charge. The zeroth components S_0^k represent the strong-charge potentials, and the spatial components $\vec{S}^k = (S_1^k, S_2^k, S_3^k)$ represent strong-rotational potentials. Hence

$$(6.3) \quad F_{SE}^k = -N g_s \nabla S_0^k$$

is defined to be the k -th component of force acting on particles with N strong charges g_s , generated by exchanging the k -th gluon. The total acting force is defined as

$$(6.4) \quad F_{SE} = -N g_s \nabla S_0, \quad S_0 = \alpha_s^k S_0^k,$$

where $\{\alpha_k^s\} = \{\alpha_s^k\}$ is the $SU(3)$ dimensionless constant vector. Also

$$(6.5) \quad \begin{aligned} F_{SM}^k &= g_s f^{kij} \vec{J}^i \times \text{curl} \vec{S}^j, \\ F_{SM} &= g_s f^{kij} \alpha_s^k \vec{J}^i \times \text{curl} \vec{S}^j, \end{aligned}$$

are called the strong-rotational forces, where $\vec{J}^k = (J_1^k, J_2^k, J_3^k)$ is the strong charge current density. It is clear that F_{SE} and F_{SM} in (6.4) and (6.5) obey PRI.

In particular, for quarks $J_\mu^k = Q_\mu^k$ are as in (5.33), and

$$J_0^k = \bar{q} \gamma_0 \lambda^k q$$

represents strong charge density of quarks.

6.2. Quark potentials. We know that the mediators of strong interaction are the eight gluons g_k ($1 \leq k \leq 8$), with corresponding gauge vector fields S_μ^k :

$$\{S_\mu^k = (S_0^k, S_1^k, S_2^k, S_3^k)\} \longleftrightarrow \{g_k\}.$$

As addressed earlier, for each k , the component S_0^k represents the k -th component of the quark potential. Namely, the k -th component of quark force F^k is given by

$$F^k = -\nabla \Phi^k, \quad \Phi^k = g_s S_0^k.$$

We now derive an approximate formula for the quark potentials Φ^k from the field equations (5.31) and (5.32). For simplicity, we only consider the case ignoring the coupling of ϕ^s with A_μ, W_μ^a and S_μ^k , as gluons are massless bosonic fields. Thus using the duality (5.4), the field equations (5.35) are written as

$$(6.6) \quad \partial^\nu S_{\nu\mu}^k - \frac{g_s}{2\hbar c} f^{kjl} g^{\alpha\beta} S_{\alpha\mu}^j S_\beta^l - g_s Q_\mu^k = \frac{g_s \zeta^k}{\sqrt{\hbar c}} \left[\partial_\mu + \frac{k_0^2}{4} x_\mu \right] \phi^s,$$

where $k_0 = m_\pi c / \hbar$. Equations (6.6) need to be supplemented with a coupling gauge equation for compensating ϕ^s generated:

$$(6.7) \quad f^{kij} \zeta^k g^{\alpha\beta} \partial^\mu (S_{\alpha\mu}^i S_\beta^j) = 0.$$

Note that

$$\partial^\mu \partial^\nu S_{\mu\nu}^k = 0 \quad \forall 1 \leq k \leq 8.$$

Taking divergence on both sides of (6.6), and making the contraction with $\{\zeta^k\}$, we have

$$(6.8) \quad \square \phi^s + k_0^2 \phi^s + k_0^2 x_\mu \partial^\mu \phi^s = -\frac{\sqrt{\hbar c} \zeta^k}{|\zeta|^2} \partial^\mu Q_\mu^k,$$

where \square is the wave operator.

By (5.33) we have

$$\partial^\mu Q_\mu^k = \partial_\mu \bar{q} \gamma^\mu \lambda_k q + \bar{q} \gamma^\mu \lambda_k \partial_\mu q.$$

In view of the Dirac equation (5.32),

$$\begin{aligned} \partial_\mu (\bar{q} \gamma^\mu \lambda_k q) &= i \frac{g_s}{\hbar c} S_\mu^j \bar{q} \gamma^\mu \lambda_j \lambda_k q + i \frac{m_g c}{\hbar} \bar{q} \lambda_k q, \\ \bar{q} \gamma^\mu \lambda_k \partial_\mu q &= -i \frac{g_s}{\hbar c} S_\mu^j \bar{q} \gamma^\mu \lambda_k \lambda_j q - i \frac{m_g c}{\hbar} \bar{q} \lambda_k q. \end{aligned}$$

Hence we arrive at

$$\partial^\mu Q_\mu^k = i \frac{g_s}{\hbar c} S_\mu^j \bar{q} \gamma^\mu [\lambda_j, \lambda_k] q = -\frac{2g_s}{\hbar c} f^{jkl} S_\mu^j Q_\mu^l,$$

where $Q^{\mu l} = g^{\mu\alpha} Q_\alpha^l$.

For a static quark, its strong charge 4-current density θ_μ and fields ϕ^s, S_μ^k satisfy

$$(6.9) \quad Q_\mu^k = \alpha_s^k \theta_\mu \delta(r), \quad \frac{\partial \phi^s}{\partial t} = \frac{\partial S_\mu^k}{\partial t} = 0.$$

Therefore, by (6.7) and (6.9), equation (6.8) is rewritten as

$$(6.10) \quad -\nabla^2 \phi^s + k_0^2 \phi^s = g_s \kappa \delta(r) - k_0^2 \vec{x} \cdot \nabla \phi^s,$$

where

$$\vec{x} \cdot \nabla \phi^s = x_1 \frac{\partial \phi^s}{\partial x_1} + x_2 \frac{\partial \phi^s}{\partial x_2} + x_3 \frac{\partial \phi^s}{\partial x_3}, \quad \kappa = \frac{2\theta_0}{\sqrt{\hbar c}} f^{kij} \frac{\theta_\mu \zeta^k}{\theta_0 |\zeta|^2},$$

and $\bar{S}_\mu^k \cong S_\mu^k(0)$ is the average

$$(6.11) \quad \bar{S}_\mu^k = \frac{1}{|B_{\rho_0}|} \int_{B_{\rho_0}} S_\mu^k dv,$$

where ρ_0 is the effective radius of quarks. Later, we shall see that

$$S_\mu^k \sim \frac{1}{r} \text{ as } r \rightarrow \infty.$$

Hence, by (6.11) we obtain

$$\bar{S}_\mu^k = \xi_\mu^k \rho_0^{-1}.$$

Thus the parameter κ is

$$(6.12) \quad \kappa = \frac{2\theta_0 D}{\sqrt{\hbar c}} \frac{1}{\rho_0}, \quad D = f^{ijk} \alpha_s^i \xi_\mu^j \frac{\theta_\mu \zeta^k}{\theta_0 |\zeta|^2}.$$

Since the quark radius $\rho_0 \cong 0$, (6.12) shows that the parameter κ is very large. Consequently equation (6.10) can be taken approximately as

$$(6.13) \quad -\nabla^2 \phi^s + k_0^2 \phi^s = g_s \kappa \delta(r) - k_0^2 \vec{x} \cdot \nabla \phi_0,$$

where ϕ_0 is the solution of the following equation

$$(6.14) \quad -\nabla^2 \phi_0 + k_0^2 \phi_0 = g_s \kappa \delta(r).$$

It is known that the solution ϕ_0 of (6.14) is given by

$$\phi_0 = \frac{g_s \kappa}{r} e^{-k_0 r}, \quad k_0 = \frac{mc}{\hbar},$$

where m is the mass of the strong dual scalar field ϕ^s —the Higgs spin-0 type boson particle with $m \geq 100 \text{ GeV}/c^2$. Physically, for the quark fields we have

$$r \leq \frac{1}{k_0} \leq 10^{-16} \text{ cm}.$$

Hence, inserting ϕ_0 in (6.13) we derive an exact solution of (6.13) as follows

$$(6.15) \quad \phi^s = g_s \kappa e^{-k_0 r} \left(\frac{1}{r} + \frac{3k_0}{4} + \frac{k_0^2}{4} r \right),$$

which is an approximate solution for (6.10).

We now return to the zeroth-components of the field equations (6.6). By (6.9), using the fact that

$$f^{klr} f^{kij} S_0^r S_0^i S_0^j = 0,$$

we derive that

$$(6.16) \quad -\nabla^2 S_0^k - \frac{3g_s}{2\hbar c} f^{klr} \nabla S_0^l \cdot \vec{S}^r + \frac{1}{2} \left(\frac{g_s}{\hbar c} \right)^2 f^{klr} f^{lij} \vec{S}^r \cdot \vec{S}^i S_0^j \\ + \frac{g_s}{\hbar c} f^{kij} \operatorname{div} \vec{S}^i S_0^j = g_s \alpha_s^k \theta_0 \delta(\vec{x}) + \frac{g_s \zeta^k}{4\sqrt{\hbar c}} k_0^2 c \tau \phi^s,$$

where τ is the lifetime of ϕ^s , and

$$\vec{S}^k = (S_1^k, S_2^k, S_3^k), \quad \operatorname{div} \vec{S}^k = \frac{\partial S_1^k}{\partial x_1} + \frac{\partial S_2^k}{\partial x_2} + \frac{\partial S_3^k}{\partial x_3}, \\ \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right), \quad \vec{x} = (x_1, x_2, x_3).$$

The equations (6.6) with $\mu \neq 0$ are given by

$$(6.17) \quad -\nabla^2 S_\mu^k - \nabla(\operatorname{div} \vec{S}^k) + \frac{g_s}{\hbar c} f^{kij} \operatorname{div}(\vec{S}^i S_\mu^j) - \frac{g_s}{2\hbar c} f^{kij} g^{\alpha\beta} S_{\alpha\mu}^i S_\beta^j \\ = \alpha_s^k \theta_\mu g_s \delta(\vec{x}) + \left[\partial_\mu + \frac{k_0^2}{4} x_\mu \right] \phi^s.$$

Physically, we have the relations

$$(6.18) \quad \theta_\mu \theta_\mu \ll \theta_0^2 \quad \text{for } 1 \leq \mu \leq 3, \\ \frac{1}{k_0} \ll c\tau \quad \text{or more precisely } k_0 c\tau > 10^5.$$

It follows from (6.17) and (6.18) that

$$(6.19) \quad |S_\mu^k S_\nu^k| \ll |S_0^k S_0^k| \quad \text{for } 1 \leq \mu, \nu \leq 3.$$

In fact, S_μ^k ($1 \leq \mu \leq 3$) represent the strong-rotational potential caused by the quark spin, and S_0^k represents the strong-charge potential generated by the charge g_s . Therefore, the property (6.19) is natural in physics, and the coupling energy of the strong-charge S_0^k and the strong-rotationsl \vec{S}^k of a quark is weak. Hence in (6.16) we have

$$(6.20) \quad f^{kij} \alpha_s^k \operatorname{div} \vec{S}^i S_0^j \cong 0, \\ f^{kij} \alpha_s^k \nabla S_0^i \cdot \vec{S}^r \cong 0, \\ f^{klr} f^{kij} \alpha_s^l \vec{S}^r \cdot \vec{S}^j S_0^i \cong 0.$$

Making the contraction for (6.16) with $\{\alpha_s^k\}$, by (6.20) and $\alpha_s^k \alpha_s^k = 1$, we deduce that

$$(6.21) \quad -\nabla^2 S_0 = g_s \theta_0 \delta(r) + \frac{g_s \zeta^k \alpha_s^k}{4\sqrt{\hbar c}} k_0^2 c \tau \phi^s,$$

where ϕ^s is given by (6.15), and

$$S_0 = S_0^k \alpha_s^k$$

is the total strong-charge potential of a quark, which yields a force exerted on particles with charge $N g_s$ as

$$F = -N g_s \nabla S_0.$$

To solve (6.21), we take S_0 in the form

$$(6.22) \quad S_0 = \frac{g_s \theta_0}{r} - \Phi.$$

Let Φ be radial symmetric, then

$$\nabla^2 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right).$$

Inserting (6.22) in (6.21) we obtain that

$$(6.23) \quad \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \Phi = \theta_0 B \rho_0^{-1} k_0^2 e^{-k_0 r} g_s \left(\frac{1}{r} + \frac{3}{4} k_0 + \frac{k_0^2}{4} r \right),$$

where

$$B = \frac{A g_s}{\sqrt{\hbar c}} c \tau, \quad k_0 = \frac{m c}{\hbar}, \quad A = \frac{\zeta^k \alpha_s^k D}{4 \sqrt{\hbar c}},$$

and D is the constant given by (6.12). Let

$$(6.24) \quad \Phi = \theta_0 g_s B \rho_0^{-1} k_0^2 e^{-k_0 r} \varphi.$$

Then, by (6.23) we deduce that

$$(6.25) \quad \varphi'' + 2 \left(\frac{1}{r} - k_0 \right) \varphi' - \left(\frac{2k_0}{r} - k_0^2 \right) \varphi = \frac{1}{r} + \frac{3}{4} k_0 + \frac{k_0^2}{4} r.$$

Assume that the solution φ of (6.25) is

$$(6.26) \quad \varphi = \sum_{k=1}^{\infty} \alpha_k r^k.$$

Inserting φ in (6.25) and comparing the coefficients of r^k , we obtain the relations

$$(6.27) \quad \begin{aligned} \alpha_1 &= \frac{1}{2}, \\ \alpha_2 &= \frac{1}{6} \left(\frac{3}{4} k_0 + 4 k_0 \alpha_1 \right), \\ \alpha_3 &= \frac{1}{12} \left(\frac{1}{4} k_0^2 + 6 k_0 \alpha_2 - k_0^2 \alpha_1 \right), \\ \alpha_4 &= \frac{1}{20} (8 k_0 \alpha_3 - k_0^2 \alpha_2), \\ &\vdots \\ \alpha_N &= \frac{1}{N(N+1)} (2N \alpha_{N-1} - \alpha_{N-2} k_0) k_0 \quad \text{for } N \geq 4. \end{aligned}$$

Often, it is enough to take only the 2nd-order approximation of the infinite series (6.26)-(6.27):

$$(6.28) \quad \varphi(r) = \alpha_1 r + \alpha_2 r^2 = \frac{r}{2} + \frac{11 k_0}{24} r^2.$$

Thus, by (6.22) and (6.24) the solution S_0 of (6.21) is given by

$$(6.29) \quad S_0 = g_s \theta_0 \left[\frac{1}{r} - \frac{B k_0^2}{2 \rho_0} e^{-k_0 r} \varphi(r) \right].$$

For the quark case studied here, we take $\theta_0 = 1$. Hence finally we have

$$(6.30) \quad S_0 = g_s \left[\frac{1}{r} - \frac{B k_0^2}{2 \rho_0} e^{-k_0 r} \varphi(r) \right],$$

Formula (6.30) provides an approximate expression for the total strong-charge potential generated by a single quark without considering the strong-rotational effect caused by the quark spin. However, if we consider N quarks occupying a ball in space with radius ρ_1 , then the parameters θ_0 in (6.29) will have to be replaced by

$$(6.31) \quad \tilde{\theta}_0 = N \left(\frac{\rho_0}{\rho_1} \right)^3 \theta_0 = N \left(\frac{\rho_0}{\rho_1} \right)^3,$$

which will be proved in the next section, where ρ_0 is the effective radius of a quark. It is the property (6.31) that causes the short range nature of strong interaction.

Quark confinement phenomena indicates that no single quark has been found, and all quarks are grouped into two or three quarks to form mesons or baryons. Therefore formula (6.29) is applicable to describing the hadron structure. From the physical point of view, the parameter k_0 in (6.29)

$$k_0 = \frac{mc}{\hbar}$$

is determined by a strong interacting Higgs particle with mass m , whose value is estimated as

$$(6.32) \quad m \geq 100 \text{ GeV}/c^2$$

or equivalently, the radius ρ_1 of hadrons is

$$(6.33) \quad \rho_1 = \frac{1}{k_0} \leq 10^{-16} \text{ cm}.$$

Therefore we believe that in the hadron level, there should be a strong interacting Higgs boson with mass as (6.32). This Higgs field might be related with the anomalies in the LHC data related to the Higgs particle.

In the nucleon level, the mediator is the strong dual particle field ϕ_s , which is the Yukawa-like particle, considered to be the π^0 meson with mass

$$m_\pi = 135 \text{ MeV}/c^2.$$

By (6.31), we shall derive in the next section the nucleon potential as

$$(6.34) \quad S_n = N \left(\frac{\rho_0}{\rho_1} \right)^3 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_1} e^{-k_1 r} \varphi(r) \right],$$

where $N = 3$ is the quark number forming nucleons, $k_1 = m_\pi c/\hbar$, ρ_0 and ρ_1 are the radii of quark and nucleon respectively.

6.3. Quark confinement and asymptotic freedom. We assume that each quark possesses a strong charge g_s which is always positive. Then the potential energy generated by two quarks with distance r is

$$\Phi = g_s S(r),$$

where $S(r) = S_0^k(r) \theta_0^k$ is the scalar quark potential, and (6.29) is an approximate formula for the quark potential. The acting force between two quarks are

$$(6.35) \quad F = -\nabla \Phi = -g_s \frac{dS}{dr}.$$

From formula (6.29) we see that there are two different radii ρ and $\rho_1 = k_0^{-1}$, ρ is the quark radius and ρ_1 is the radius of quark acting forces. Physically they satisfy

$$(6.36) \quad \rho \ll \rho_1, \quad \rho \leq 10^{-21} \text{ cm}, \quad \rho_1 \leq 10^{-16} \text{ cm}.$$

Based on (6.29) and (6.36), we derive the diagram of quark potential Φ as shown in Figure 6.1.

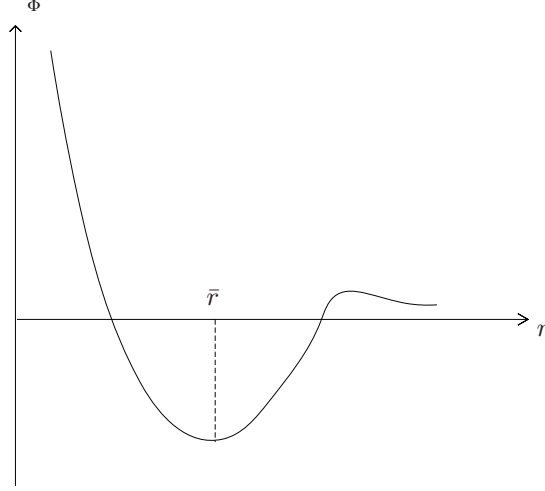


FIGURE 6.1.

Figure 6.1 shows that Φ has a minimum at \bar{r} ($\rho < \bar{r} < r_0$), where the quark acting force F is zero. Namely by (6.35) and (6.29), we have

$$(6.37) \quad F = \begin{cases} > 0 & \text{for } 0 < r < \bar{r}, \\ = 0 & \text{for } r = \bar{r}, \\ < 0 & \text{for } \bar{r} < r < r_0, \\ > 0 & \text{for } r > r_0. \end{cases}$$

We infer from (6.37) the following conclusions:

- (1) Two close enough quarks are repelling.
- (2) Near $r = \bar{r}$, there are no interactions between quarks—the interactions are weak. This explains the quark asymptotic freedom phenomena.
- (3) In the region $\bar{r} < r < r_0$, the quark acting force is attracting. In particular, the attracting potential energy has the order of magnitude as

$$\Phi \sim -\frac{c\tau}{r_0^2\rho}.$$

It implies that

$$(6.38) \quad \Phi \rightarrow -\infty \text{ as } \rho \rightarrow 0,$$

and the property (6.38) explains the quark confinement. In particular, based on (6.30) and (6.31), the ratio of binding energies of quark and nucleon is

$$(6.39) \quad \frac{E_q}{E_n} = \left(\frac{B}{\rho_0}\right) / \left(3\left(\frac{\rho_0}{\rho_1}\right)^3 \frac{B_n}{\rho_1}\right) \sim \left(\frac{\rho_1}{\rho_0}\right)^4 \sim 10^{20},$$

which is in the Planck level.

- (4) $F > 0$ as $r > r_0$ means the quark attracting force is a short range force.
- (5) The radius r_0 represents the radius of hadrons, which is estimated as $r_0 \leq 10^{-16}\text{cm}$.

7. STRONG INTERACTION POTENTIAL

7.1. QCD action for nucleons. Interaction forces act on different levels of particles/matter. Strong interaction forces are generated in three level of particles: quarks, hadrons/nucleons, and atoms. Beyond the level of atoms, the strong force almost disappears. In Section 3 we have derived the quark potential (6.29), and we devote this section to deriving hadron/nucleon and atom force potentials.

Nucleons include protons and neutrons which are the constituents of a nuclear. Classically, the force holding nucleons together to form a nuclear is the Yukawa potential

$$(7.1) \quad \Phi_Y = -\frac{g}{r}e^{-k_1 r},$$

where $k_1 = m_\pi c/\hbar$, m_π is the mass of the Yukawa meson, g is the meson charge with $g \cong 10e$, and e is the electric charge.

The Yukawa potential (7.1) is a phenomenological theory, which provides an approximation for the short range strong interaction force between nucleons. However, formula (7.1) fails to explain the repelling phenomenon as shown in Figure 6.1 when two nucleons are close.

In the same spirit as for deriving the quark potential (6.29), we now deduce (6.34) replacing (7.1). To this end, we start with the *QCD* action for nucleons as

$$(7.2) \quad \mathcal{L} = -\frac{1}{4}S_{\mu\nu}^k S^{k\mu\nu} + \bar{n}(i\gamma^\mu D_\mu - mc^2)n,$$

where $S_{\mu\nu}^k$ are as in (4.27) representing gluon fields, $n = (a_1, a_2, a_3)\tilde{n}$ with \tilde{n} being the wave function of a nucleon and $a_1^2 + a_2^2 + a_3^2 = 1$, and

$$(7.3) \quad D_\mu n = (\hbar c \partial_\mu + ig_s S_\mu^k \lambda_k)n.$$

Because the adjoint field ϕ of nucleons represents the π meson-like particle field, similar to (5.31) and (5.33), from (7.2)-(7.3), we derive the field equations describing nucleons as follows

$$(7.4) \quad \partial^\nu S_{\nu\mu}^k + \frac{g_s}{2\hbar c} f^{kij} g^{\alpha\beta} S_{\alpha\mu}^i S_\beta^j - g_s J_\mu^k = \frac{g_s \zeta^k}{\sqrt{\hbar c}} \left(\partial_\mu + \frac{k_1^2}{4} x_\mu \right) \phi,$$

$$(7.5) \quad i\gamma^\mu (\hbar c \partial_\mu + ig_s S_\mu^k \lambda_k) n - mc^2 n = 0,$$

where

$$(7.6) \quad J_\mu^k = \bar{n} \gamma_\mu \lambda^k n \quad (\lambda^k = \lambda_k),$$

$$(7.7) \quad k_1 = m_\pi c/\hbar.$$

7.2. Nucleon/hadron potential. In the same fashion as deriving (6.10), we deduce from (7.4) and (7.5) that

$$(7.8) \quad -\nabla^2 \phi + k_1^2 \phi = g_s \rho_1^{-1} A_n \tilde{\theta}_0 \delta(r) - k_0^2 \vec{x} \cdot \nabla \phi,$$

where ρ_1 is the radius of a nucleon, and

$$(7.9) \quad A_n = \frac{2}{\sqrt{\hbar c}} f^{kij} \frac{\zeta^k \tilde{\alpha}_s^i \xi_\mu^j}{|\zeta|^2} \frac{\tilde{\theta}_\mu}{\tilde{\theta}_0}.$$

Here $\tilde{\theta}_\mu$ is defined by

$$(7.10) \quad J_\mu^k = \alpha_s^k \tilde{\theta}_\mu \delta(r).$$

The total potential equation is given by

$$(7.11) \quad -\nabla^2 S_n = g_s \tilde{\theta}_0 \delta(r) + \frac{g_s \zeta^k \tilde{\alpha}_s^k}{4\sqrt{\hbar c}} k_1^2 c\tau \phi,$$

where ϕ is as in (7.8), and

$$S_n = S_0^k \alpha_0^k$$

is the total potential of a nucleon.

Similar to (6.29), the solution of (7.11) are given by

$$(7.12) \quad S_n = \tilde{\theta}_0 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_1} e^{-k_1 r} \varphi(r) \right].$$

Here $\varphi(r)$ is as (6.26) and B_n is a constant given by

$$B_n = \frac{A_n g_s \zeta^k \alpha_s^k}{4\sqrt{\hbar c}} c\tau,$$

A_n is as in (7.9), τ is the lifetime of the Yukawa particle, and k_1 is as (7.7).

By (6.9) and (7.10) we have

$$(7.13) \quad \frac{NV_q}{V_n} = \frac{|J_0|}{|Q_0|} = \frac{|\tilde{\theta}_0|}{|\theta_0|},$$

where V_n and V_q are the volumes of nucleon and quark, $|J_0| = \sqrt{J_0^k J_0^k}$, $|Q_0| = \sqrt{Q_0^k Q_0^k}$, and $N = 3$ is the number of quarks in a nucleon. By $V_g/V_n = \left(\frac{\rho_1}{\rho_0}\right)^3$, from (7.13) and $\theta_0 = 1$, we deduce that

$$\tilde{\theta}_0 = 3 \left(\frac{\rho_0}{\rho_1} \right)^3.$$

Thus (7.12) can be expressed as

$$(7.14) \quad S_n = 3 \left(\frac{\rho_0}{\rho_1} \right)^3 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_1} e^{-k_1 r} \varphi(r) \right],$$

which has the same form as (6.34).

With the same method as above, an atom/molecule with N nucleons generates the strong interaction potential as follows

$$(7.15) \quad S_a = 3N \left(\frac{\rho_0}{\rho_1} \right)^3 \left(\frac{\rho_1}{\rho_2} \right)^3 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_2} e^{-k_1 r} \varphi(r) \right],$$

where ρ_2 is the radius of an atom, and k_1 is as in (7.14).

7.3. Physical conclusions. We have derived three formulas (6.29), (7.14) and (7.15) describing three different levels of strong interaction. The potential (6.29) reveals the hadron structure and explains the mechanism and mature of quark confinement and asymptotic freedom. Hereafter we shall see that formula (7.14) agrees with the observed data for nucleons/hadrons, and (7.15) can explain why the strong forces disappear in the macro-scale (short-range nature of the strong interaction).

We know that

$$\rho_1 \leq 10^{-16} \text{cm}, \quad k_1 = 10^{13} \text{cm}^{-1}, \quad r_1 = \frac{1}{k_1} = 1f_m.$$

For the polynomial φ in (6.26)-(6.27), we take the first-order approximation

$$\varphi = \frac{r}{2}.$$

Then (7.14) reads as

$$(7.16) \quad S_n = 3g_s \left(\frac{\rho_0}{\rho_1} \right)^3 \left[\frac{1}{r} - \frac{10^{16}}{2} \frac{B_n}{r_1^2} e^{-\frac{r}{r_1}} r \right].$$

The force acting on one nucleon by another is

$$(7.17) \quad \begin{aligned} F &= -3g_s \frac{dS_n}{dr} \\ &= 9g_s^2 \left(\frac{\rho_0}{\rho_1} \right)^3 \left[\frac{1}{r^2} - \frac{10^{16}}{2} \cdot \frac{B_n}{r_1^2} e^{-\frac{r}{r_1}} \left(\frac{r}{r_1} - 1 \right) \right] \\ &= 9g_s^2 \left(\frac{\rho_0}{\rho_1} \right)^3 \frac{1}{r^2} - \frac{G_n}{r_1^2} \left(\frac{1}{r_1} - 1 \right) e^{-\frac{r}{r_1}}, \end{aligned}$$

where

$$G_n = \frac{9}{2} \times 10^{16} \times \left(\frac{\rho_0}{\rho_1} \right)^3 B_n g_s^2.$$

With (7.1), the Yukawa force is given by

$$(7.18) \quad F_Y = g^2 \frac{d}{dr} \left(\frac{1}{r} e^{-\frac{r}{r_1}} \right) = -g^2 \left(\frac{1}{r^2} + \frac{1}{r_1 r} \right) e^{-\frac{r}{r_1}}.$$

Comparing (7.17) with (7.18), we may take

$$(7.19) \quad g_s = g^2.$$

Namely,

$$\frac{9}{2} \times 10^{16} \times \left(\frac{\rho_0}{\rho_1} \right)^3 B_n \sim 2.$$

We derive from (7.16)-(7.19) the following conclusions, consistent with experimental results:

- (1) The diagram of the nucleon/hadron potential (7.16) is as shown in Figure 6.1.
- (2) By (7.17), nucleons have a repelling radius

$$a \cong 1f_m,$$

and the repelling force F tends to infinite as $r \rightarrow 0$:

$$F \rightarrow +\infty \quad \text{as } r \rightarrow 0.$$

- (3) There exists an attracting region:

$$1f_m < r < zf_m,$$

where z satisfies that

$$z^2 e^{-z} (z - 1) = 2 \times 10^{-16} B_n^{-1}.$$

Hence $z = 30 \sim 40$.

(4) It is known that the radius of an atom is about

$$\rho_2 \cong 10^{-8} \text{ cm.}$$

and

$$\left(\frac{\rho_1}{\rho_2}\right)^3 \leq 10^{-24}.$$

In addition, the gravity and the Yukawa force are

$$(7.20) \quad \frac{Gm_p^2}{\hbar c} \sim 10^{-38}, \quad \frac{g^2}{\hbar c} \sim 10.$$

Hence by (7.19) and (7.20), beyond the level of an atom or a molecule, the ratio between the strong repelling force and the gravitational force is

$$(7.21) \quad \frac{F_s}{F_g} = \left(3N^2 \left(\frac{\rho_0}{\rho_2}\right)^3 g_s^2\right) / (N^2 Gm_p^2) = 3 \times 10^{39} \left(\frac{\rho_0}{\rho_2}\right)^3.$$

Physically, the effective quark radius is taken as $\rho \sim 10^{-21} \text{ cm}$, and the atom or molecule radius is $\rho_2 = 10^{-8} \text{ cm}$ or $\rho_2 = 10^{-7} \text{ cm}$. Then it follows from (7.21) that

$$\begin{aligned} \frac{F_s}{F_g} &\sim 3 && \text{near the atom radius } \rho_a, \\ \frac{F_s}{F_g} &\sim 3 \times 10^{-3} && \text{beyond the molecule radius } \rho_m. \end{aligned}$$

Namely, near the radius of an atom, the strong repelling is stronger than the gravitational force, and beyond the molecule radius, the strong repelling force is smaller than the gravitational force. We believe this competition between the gravitational force and the strong force in the level of atoms/molecules gives rise to the mechanism of the van der Waals force.

8. DUALITY THEORY OF WEAK INTERACTIONS

8.1. Non-coexistence of charged and neutral particles. In Section 10, we will discuss the mass generation mechanism for the field equations (5.22) and (5.23). We focus here on charged Higgs particles and the non-coexistence of weak interaction intermediate vector bosons using these field equations.

Equation (5.22) need to be supplemented with coupling gauge equations to complement the adjoint fields ϕ_a created, which are taken as

$$(8.1) \quad \partial^\mu W_\mu^a = \beta^a \quad \text{for } a = 1, 2, 3,$$

where β^a are parameters which may vary for different physical situations.

For simplicity we take the Pauli matrices σ_a ($1 \leq a \leq 3$) as the generators of $SU(2)$. Then make the transformation

$$(8.2) \quad \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}.$$

Under this transformation, $(W_\mu^1, W_\mu^2, W_\mu^3)$ and (ϕ_1, ϕ_2, ϕ_3) are transformed to

$$\begin{aligned} (W_\mu^\pm, Z_\mu) &= (W_\mu^1 \pm iW_\mu^2, W_\mu^3), \\ (\phi^\pm, \phi^0) &= (\phi_1 \pm i\phi_2, \phi_3). \end{aligned}$$

Then by PRI, equations (5.22) become

$$(8.3) \quad \begin{aligned} \partial^\nu W_{\nu\mu}^\pm \pm \frac{ig_w}{2\hbar c} g^{\nu\nu} (W_{\nu\mu}^\pm Z_\nu - Z_{\nu\mu} W_\nu^\pm) - g_w J_\mu^\pm \\ = \left[\partial_\mu - k_W^2 W_\mu^\pm - k_Z^2 Z_\mu + \frac{k_0^2}{4} x_\mu \right] \phi^\pm, \end{aligned}$$

$$(8.4) \quad \begin{aligned} \partial^\nu Z_{\nu\mu} - \frac{ig_w}{2\hbar c} g^{\nu\nu} (W_{\nu\mu}^+ W_\nu^- - W_{\nu\mu}^- W_\nu^+) - g_w J_\mu^0 \\ = \left[\partial_\mu - k_W^2 W_\mu^\pm - k_Z^2 Z_\mu + \frac{k_0^2}{4} x_\mu \right] \phi^0, \end{aligned}$$

where

$$(8.5) \quad \begin{aligned} J_\mu^\pm &= \frac{1}{\sqrt{2}} (J_\mu^1 \pm iJ_\mu^2), \quad J_\mu^{NC} = J_\mu^3, \\ W_{\nu\mu}^\pm &= \partial_\nu W_\mu^\pm - \partial_\mu W_\nu^\pm \pm \frac{ig_w}{\hbar c} (Z_\mu W_\nu^\pm - Z_\nu W_\mu^\pm), \\ Z_{\nu\mu} &= \partial_\nu Z_\mu - \partial_\mu Z_\nu + \frac{ig_w}{\hbar c} (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+). \end{aligned}$$

Here

$$(8.6) \quad k_W^2 = \frac{g_w \alpha_1^w}{\sqrt{2}\hbar c}, \quad k_Z^2 = \frac{g_w \alpha_3^w}{\hbar c}, \quad \begin{pmatrix} k_W^2 \\ k_W^2 \\ k_Z^2 \end{pmatrix} = \frac{g_w}{\sqrt{\hbar c}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1^w \\ \alpha_2^w \\ \alpha_3^w \end{pmatrix},$$

where $\{\alpha_b^w\} = (\alpha_1^w, \alpha_2^w, \alpha_3^w)$ is as in (5.22), and the second component $\alpha_2^w = 0$ when we use the Pauli representation.

It is easy to see that (8.3) for W_μ^+ and W_μ^- are complex conjugate to each other. Here are two important solutions, leading to two different weak interactions:

First, if

$$(8.7) \quad W_\mu^\pm = 0, \quad \phi^0 = 1, \quad \beta^a = 0,$$

then Z_μ satisfies the equation

$$(8.8) \quad \square Z_\mu + k_Z^2 Z_\mu - g J_\mu^0 - \frac{k_0^2}{4} x_\mu = 0$$

where \square is the wave operator given by

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2.$$

This is the case where the weak interaction involves the neutral Higgs boson ϕ^0 and the neutral intermediate vector boson Z with mass parameter k_Z^2 .

Second, if

$$(8.9) \quad Z_\mu = 0, \quad \phi^\pm = 1, \quad \beta^a = 0,$$

then W_μ^\pm satisfy

$$(8.10) \quad \square W_\mu^\pm + k_W^2 W_\mu^\pm - g J_\mu^\pm - \frac{k_0^2}{4} x_\mu = o(W^\pm)$$

This is the case where the weak interaction occurs through the two charged intermediate vector bosons W^\pm , with mass parameter k_W^2 , and the two charged Higgs bosons ϕ^\pm .

These two solution cases suggest that the charged gauge bosons W^\pm cannot appear simultaneously with the neutral boson Z in one physical situation.

Now we consider the adjoint fields ϕ^\pm and ϕ^0 . If

$$(8.11) \quad Z_\mu = 0, \quad \beta^1 < 0, \quad \beta^2 = 0,$$

taking divergence on both sides of (8.3) we get

$$(8.12) \quad \square\phi^\pm + (k_0^2 + k_W^2|\beta^1|)\phi^\pm + g_w\partial^\mu J_\mu^\pm = o(W^\pm, \phi^\pm).$$

Also, if

$$(8.13) \quad W_\mu^\pm = 0, \quad \beta^3 < 0,$$

then we obtain from (8.4) that

$$(8.14) \quad \square\phi^0 + (k_0^2 + k_Z^2|\beta^3|)\phi^0 + g\partial^\mu J_\mu^0 = o(Z, \phi^0).$$

Hence these two cases suggest also that there exist charged and neutral Higgs particles ϕ^\pm and ϕ^0 , and the charged Higgs ϕ^\pm cannot coexist with the neutral Higgs ϕ^0 .

In summary, from the above discussion we deduce the following physical conclusions:

- 1) Existence of charged and neutral Higgs particles ϕ^\pm and ϕ^0 , satisfying equations (8.12) and (8.14) respectively.
- 2) Non-coexistence of charged and neutral weak interaction particles. Namely, W^\pm and ϕ^\pm cannot coexist with Z or ϕ^0 .
- 3) Finally, the two parameters $k_0^2 + k_W^2|\beta^1|$ and $k_0^2 + k_Z^2|\beta^3|$ define the masses m_H^c and m_H^0 of the Higgs bosons ϕ^\pm and ϕ^0 . We have

$$\frac{(m_H^c)^2}{(m_H^0)^2} = \frac{k_0^2 + k_W^2|\beta^1|}{k_0^2 + k_Z^2|\beta^3|} = \frac{m_0^2 + m_W^2|\beta^1|}{m_0^2 + m_Z^2|\beta^3|},$$

where m_0 is the mass associated with the mass potential k_0 . We conjecture that the masses m_H^c and m_H^0 of ϕ^\pm and ϕ^0 also satisfy the scale relation (8.19), i.e.

$$(8.15) \quad \frac{m_H^c}{m_H^0} = \frac{m_W}{m_Z} = \frac{|j^\pm|}{|j_\mu^{NC}|} = \cos\theta_W.$$

where θ_W is the Weinberg angle.

We remark that Conclusions 1) and 2) above cannot be derived from the classical weak interaction theories.

8.2. Scaling relation. We know from (10.36) that

$$(8.16) \quad \frac{m_W}{m_Z} = \cos\theta_W.$$

According to the IVB theory for weak interaction, the charged and the neutral currents are

$$(8.17) \quad j_\mu^\pm = \frac{g_w}{\sqrt{2}}J_\mu^\pm, \quad j_\mu^{NC} = \frac{g_w}{\cos\theta_W}J_\mu^{NC}.$$

After proper scaling for J_μ^\pm , i.e. taking $\frac{1}{\sqrt{2}}J_\mu^\pm$ as J_μ^\pm , (8.17) can be rewritten as

$$j_\mu^\pm = g_w J_\mu^\pm, \quad j_\mu^{NC} = \frac{g_w}{\cos\theta_W} J_\mu^{NC},$$

Hence we can consider g_w and $g_w/\cos\theta_W$ as the intensities of the currents j_μ^\pm and j_μ^{NC} respectively, denoted by

$$(8.18) \quad |j_\mu^\pm| = g_w, \quad |j_\mu^{NC}| = \frac{g_w}{\cos\theta_W}.$$

Therefore, from (8.16) and (8.18) we get the scale relation between masses and intensities of currents as

$$(8.19) \quad \frac{|j_\mu^\pm|}{|j_\mu^{NC}|} = \frac{m_{W^\pm}}{m_Z} = \cos\theta_W.$$

By PRI, the weak interaction can be decoupled with other interactions. If we use the Pauli matrices σ_1, σ_2 and σ_3 as the generators for $SU(2)$, then G_{ab} is Euclidean. However, the corresponding action density \mathcal{L}_W does not lead to the scaling relation (8.19). To solve this problem, we take another $SU(2)$ representation with the following generators:

$$(8.20) \quad \tau_1 = \sigma_1, \quad \tau_2 = \sigma_2, \quad \tau_3 = \sqrt{\cos\theta_W}\sigma_3.$$

In this case, the metric G_{ab} defined by (3.19) is

$$G_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta_W \end{pmatrix}.$$

By (3.20) the action density corresponding to the representation (8.20) is given by

$$(8.21) \quad \begin{aligned} \mathcal{L}_W = & -\frac{1}{4}[W_{\mu\nu}^1 W^{1\mu\nu} + W_{\mu\nu}^2 W^{2\mu\nu} + \cos\theta_W W_{\mu\nu}^3 W^{3\mu\nu}] \\ & + \bar{L}[i\gamma^\mu(\partial_\mu - ig_w W_{\mu\nu}^a \tau_a) - m^L]L, \end{aligned}$$

where

$$(8.22) \quad W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \frac{g_w}{\hbar c} \lambda_{bc}^a W_\mu^b W_\nu^c,$$

and λ_{bc}^a are the structural constants with respect to (8.20), which are antisymmetric for all indices a, b, c .

Thus, under the div_A -free constraint associated with

$$\begin{aligned} D_{\mu 1} &= \partial_\mu - \left(\frac{m_W c}{\hbar}\right)^2 W_\mu^1 + \frac{k_0^2}{4} x_\mu, \\ D_{\mu 2} &= \partial_\mu - \left(\frac{m_W c}{\hbar}\right)^2 W_\mu^2 + \frac{k_0^2}{4} x_\mu, \\ D_{\mu 3} &= \cos\theta_W \partial_\mu - \frac{1}{\cos\theta_W} \left(\frac{m_W c}{\hbar}\right)^2 W_\mu^3 + \cos\theta_W \frac{k_0^2}{4} x_\mu, \end{aligned}$$

the Euler-Lagrangian equations of (8.21)-(8.22) are as follows

$$\begin{aligned}
(8.23) \quad & \partial^\nu W_{\nu\mu}^1 + k^2 \phi W_\mu^1 - \frac{g_w g^{\nu\nu}}{2\hbar c} (W_{\nu\mu}^2 W_\nu^3 - W_{\nu\mu}^3 W_\nu^2) \\
& - g_w J_\mu^1 = \left[\partial_\mu + \frac{k_0^2}{4} x_\mu \right] \phi, \\
& \partial^\nu W_{\nu\mu}^2 + k^2 \phi W_\mu^2 - \frac{g_w g^{\nu\nu}}{2\hbar c} (W_{\nu\mu}^3 W_\nu^1 - W_{\nu\mu}^1 W_\nu^3) \\
& - g_w J_\mu^2 = \left[\partial_\mu + \frac{k_0^2}{4} x_\mu \right] \phi, \\
& \partial^\nu W_{\nu\mu}^3 + \frac{k^2}{\cos^2 \theta_W} \phi W_\mu^3 - \frac{g_w}{2\hbar c \cos \theta_W} g^{\nu\nu} (W_{\nu\mu}^1 W_\nu^2 - W_{\nu\mu}^2 W_\nu^1) \\
& - \frac{g_w}{\cos \theta_W} J_\mu^3 = \left[\partial_\mu + \frac{k_0^2}{4} x_\mu \right] \phi,
\end{aligned}$$

where $k = m_W c / \hbar$. Under the unitary rotation transformation

$$(8.24) \quad \begin{pmatrix} W_\mu^+ \\ W_\mu^- \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \end{pmatrix}.$$

By PRI, the equations (8.23) becomes

$$\begin{aligned}
(8.25) \quad & \partial^\nu W_{\nu\mu}^\pm + k^2 \phi W_\mu^\pm \pm \frac{i g_w}{2\hbar c} g^{\nu\nu} (W_{\nu\mu}^\pm Z_\nu - Z_{\nu\mu} W_\nu^\pm) \\
& - g_w J_\mu^\pm = \eta^\pm \left[\partial_\mu + \frac{k_0^2}{4} x_\mu \right] \phi, \\
& \partial^\nu Z_{\nu\mu} + \frac{k^2}{\cos^2 \theta_W} \phi Z_\mu - \frac{i g_w}{2\hbar c \cos \theta_W} g^{\nu\nu} (W_{\nu\mu}^+ W_\nu^- - W_{\nu\mu}^- W_\nu^+) \\
& - \frac{g_w}{\cos \theta_W} J_\mu^{NC} = \left[\partial_\mu + \frac{k_0^2}{4} x_\mu \right] \phi,
\end{aligned}$$

where J_μ^\pm , $J_\mu^{NC} = J_\mu^3$ and $W_{\nu\mu}^\pm$, $Z_{\nu\mu}$ are defined (8.5). It is clear that for (8.25), scaling relation (8.19) holds true.

Note here that the 2nd-order $SU(2)$ tensor diag $(k^2, k^2, k^2 / \cos^2 \theta_W)$ are invariant for the transformation (8.24). Namely

$$\begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & \frac{k^2}{\cos^2 \theta_W} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & \frac{k^2}{\cos^2 \theta_W} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^\dagger.$$

Hence from (8.23) to (8.25) we have

$$\left(k^2 \phi W_\mu^1, k^2 \phi W_\mu^2, \frac{k^2}{\cos^2 \theta_W} \phi W_\mu^3 \right) \longrightarrow \left(k^2 \phi W_\mu^+, k^2 \phi W_\mu^-, \frac{k^2 \phi}{\cos^2 \theta_W} Z_\mu \right).$$

Remark 8.1. In (8.19), the ratio between the mass loss and intensity loss of the charged bosons W^\pm and charged currents j^\pm is the same as the ratio between those of Z and j^{NC} . The parts lost can be considered as being transformed into electromagnetic energy.

9. WEAK INTERACTION POTENTIALS

9.1. Weak interaction potentials. We now consider the duality between $\{W_\mu^a\}$ and a single neutral Higgs field given by (5.26). It is clear that both the weak gauge fields W_μ^a and the adjoint scalar field ϕ carry rich physical information, as the electromagnetic potential A_μ in QED. For example, the electric field E and magnetic field H are written as

$$E = - \left(\frac{\partial \vec{A}}{\partial x^0} + \nabla A_0 \right), \quad H = \text{curl} \vec{A},$$

where $\vec{A} = (A_1, A_2, A_3)$, $\nabla = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$, the electromagnetic energy density ε is

$$\varepsilon = \frac{1}{8\pi}(E^2 + H^2),$$

and the photon γ is expressed by A_μ satisfying

$$\square A_\mu = 0.$$

So far, very little information has been extrapolated from the weak gauge fields. For example, we know that Z_μ and W_μ^\pm satisfying (8.8) and (8.10) represent the neutral and charged bosons, and ϕ^\pm, ϕ^0 satisfying (8.12) and (8.14) represent the neutral and charged Higgs particles.

In the same spirit as electromagnetism, we introduce below two physical quantities associated with the weak gauge potentials W_μ^a .

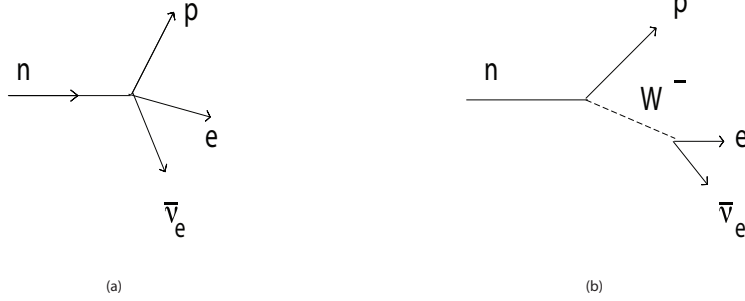


FIGURE 9.1.

First, the β -decay is a weak process, as illustrated by Figure 9.1 (a) and (b). Physically, the process in Figure 9.1 (a) is regarded as an exchange of a massive

vector meson W^- , as shown in (b). The force range is about $r = 10^{-16}$ cm. Before the β -decay, the neutron n is an energy pack bound by the potential energy ϕ in the radius $r = 10^{-16}$, and when the momentum energy in the interior of a neutron is greater than the bounding energy, the neutron is split into a proton p and an intermediate vector boson W^- , and the β -decay occurs. The interior momentum energy is characterized by

$$(9.1) \quad M = \int G_{ab} \nabla W_\mu^a \nabla W_\mu^b dy, \quad y \in \mathbb{R}^3,$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$. Obviously, the right-hand side of (9.1) obeys PRI. Since ε is the momentum energy, it is not Lorentz invariant.

Second, the weak gauge potential W_μ^a have three constituents:

$$\{W_\mu^1, W_\mu^2, W_\mu^3\}.$$

The time-components W_0^a represent the weak-charge potentials with corresponding forces exerted by a particle with one weak charge g_w on another with N weak charges given by:

$$F_{WE}^a = -Ng_w \nabla W_0^a \quad \text{for } a = 1, 2, 3.$$

The total force exerted on the particle is

$$(9.2) \quad F_{WE} = -Ng_w \alpha_a^w \nabla W_0^a,$$

where α_a^w is as in (5.22).

The spatial components $\vec{W}^a = (W_1^a, W_2^a, W_3^a)$ represent the weak-rotational potentials, yielding the following weak-rotational forces:

$$(9.3) \quad \begin{aligned} F_{WM}^a &= g_w \varepsilon^{abc} \vec{J}^b \times \text{curl} \vec{W}^c, \\ F_{WM} &= g_w \varepsilon^{abc} \alpha_a^w \vec{J}^b \times \text{curl} \vec{W}^c, \end{aligned}$$

where $\{\vec{J}^b\} = \{J_1^b, J_2^b, J_3^b\}$ is the weak current density. Obviously, F_{WE} and F_{WM} are gauge group representation invariant, i.e. they obey PRI.

9.2. Dual field potential. We take the Pauli matrices σ_a as the generators of an $SU(2)$ representation. Thus, $G_{ab} = \delta_{ab}$ and we derive from (5.27) the following equation

$$(9.4) \quad \begin{aligned} \square \phi + \left(\frac{m_H c}{\hbar} \right)^2 \phi + \frac{\xi^a}{\sqrt{\hbar c}} \alpha_b^w W_\mu^b D^\mu \phi \\ = -\sqrt{\hbar c} \xi^a \left[\partial^\mu J_{\mu a} + \frac{g_w}{2\hbar c} \lambda_{cd}^b g^{\alpha\beta} \partial^\mu (W_{\alpha\nu}^c W_\beta^d) \right], \end{aligned}$$

where $\xi^a = \zeta^a / |\zeta|^2$ ($\zeta^a = \zeta_a$).

Assume that ϕ and W_μ^a are small and are independent of time variable $x_0 = ct$. Ignoring the higher order terms, equation (9.4) becomes

$$(9.5) \quad \nabla^2 \phi - \left(\frac{m_H c}{\hbar} \right)^2 \phi = -\sqrt{\hbar c} \xi^a \partial^\mu J_{\mu a}.$$

Equation (9.5) provides a model describing the scalar potential ϕ of weak force, which holds energy to form a particle.

By definition, we have

$$\partial^\mu J_{\mu a} = \partial_\mu \bar{L} \gamma^\mu \sigma_a L + \bar{L} \gamma^\mu \sigma_a \partial_\mu L.$$

By the Dirac equation (5.23).

$$\begin{aligned}\partial_\mu \bar{L} \gamma^\mu \sigma_a L &= -ig_w W_\mu^b \bar{L} \gamma^\mu \sigma_b \sigma_a L + im \bar{L} \sigma_a L, \\ \bar{L} \gamma^\mu \sigma_a \partial_\mu L &= ig_w W_\mu^b \bar{L} \gamma^\mu \sigma_a \sigma_b L - im \bar{L} \sigma_a L.\end{aligned}$$

Thus we obtain

$$\partial^\mu J_{\mu a} = ig_w W_\mu^b \bar{L} \gamma^\mu [\sigma_a, \sigma_b] L = -2g_w \varepsilon_{abc} W_\mu^b J_c^\mu.$$

Noting that

$$\varepsilon_{abc} \xi^a W_\mu^b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \xi^1 & \xi^2 & \xi^3 \\ W_\mu^1 & W_\mu^2 & W_\mu^3 \end{vmatrix} = \vec{\xi} \times \vec{W}_\mu$$

where $\vec{\xi} = (\xi^1, \xi^2, \xi^3)$, $\vec{W}_\mu = (W_\mu^1, W_\mu^2, W_\mu^3)$. Hence we obtain that

$$(9.6) \quad \xi^a \partial^\mu J_{\mu a} = 2g_w (\vec{\xi} \times \vec{W}_\mu) \cdot \vec{J}^\mu, \quad \vec{J}^\mu = (J_1^\mu, J_2^\mu, J_3^\mu).$$

The weak charge densities $J_0^a = J_a^0$ are

$$(9.7) \quad J_0^a = \alpha_w^a \delta(x) \quad \text{for } a = 1, 2, 3,$$

where $\vec{\alpha}_w = (\alpha_w^1, \alpha_w^2, \alpha_w^3)$ is as in (5.22). Therefore it follows from (9.6) that

$$(9.8) \quad \xi^a \partial^\mu J_{\mu a} = 2g_w \vec{\alpha}_w \cdot (\vec{\xi} \times \vec{W}_\mu) \delta(x),$$

where $\vec{\omega} = \vec{W}_0(0)$, $\vec{\xi} = (\zeta_1, \zeta_2, \zeta_3)/|\zeta|^2$, and

$$(9.9) \quad \kappa = 2\vec{\alpha}_w \cdot (\vec{\xi} \times \vec{\omega}) = 2\vec{\xi} \cdot (\vec{\omega} \times \vec{\alpha}_w) = \begin{vmatrix} \xi^1 & \xi^2 & \xi^3 \\ \omega^1 & \omega^2 & \omega^3 \\ \alpha_w^1 & \alpha_w^2 & \alpha_w^3 \end{vmatrix}.$$

Thus, by (9.8) and (9.9) the equation (9.5) is rewritten as

$$(9.10) \quad \nabla^2 \phi - \left(\frac{m_{HC}}{\hbar} \right)^2 \phi = -\kappa g_w \delta(x).$$

Let

$$(9.11) \quad k_H = \frac{m_{HC}}{\hbar},$$

where m_H is the mass of a Higgs particle.

By (9.10) we derive the dual field potential ϕ :

$$(9.12) \quad \phi = \frac{\kappa g_w}{r} e^{-k_H r}.$$

Formula (9.12) leads to a few physical conclusions for weak interaction as follows:

1). The masses m_H and m_π of the Higgs and π meson are

$$m_H \cong 125 \text{ GeV}/c^2, \quad m_\pi = 0.135 \text{ GeV}/c^2,$$

which implies that

$$m_H/m_\pi \cong 10^3.$$

By (9.12) we have

$$\frac{r_W}{r_S} = \frac{m_\pi}{m_H} = 10^{-3},$$

where r_W and r_S are the force ranges of weak and strong interactions. Hence the weak force range is $r_W = 10^{-16} \text{ cm}$, consistent with experimental data.

2). By (9.12), the $SU(2)$ coupling constant g_w in (8.22) is endowed with a new physical meaning as the weak charge, reminiscent of the electric charge e .

3). The weak force parameter κ given by (9.9) is an $SU(2)$ pseudo-scalar. In addition, since the quantities $\omega^a = W_0^a(0)$ and θ^a defined by (9.7) characterize the interior properties of weak interaction particles such as the electron e , the neutron n and the proton p , the parameter κ reflects the interior structure of e, n, p .

4). For a particle, e.g. for the neutron n , we conjecture that the condition for decay depends on if the interior momentum M defined by (9.1) satisfies the following condition

$$(9.13) \quad M \geq \int_{|y| < r_0} \frac{g_w^2}{(\hbar c)^{3/2}} \zeta^a \alpha_w^b W_\mu^a W_\mu^b \phi dy, \quad y \in \mathbb{R}^3, \quad r_0 = k_0^{-1}.$$

In this case, n decays as

$$n \rightarrow p + e + \bar{\nu}_e.$$

Otherwise, if

$$(9.14) \quad M < \int_{|y| < r_0} \frac{g_w^2}{(\hbar c)^{3/2}} \zeta^a \alpha_w^b W_\mu^a W_\mu^b \phi dy,$$

the neutron n does not decay. Hence this explains why neutrons can spontaneously undergo a β -decay under proper conditions.

5). By (9.9), κ can be expressed as

$$\kappa = |\vec{\theta}| \cdot |\vec{\omega}| \cos \Phi \sin \varphi,$$

where Φ = the angle between $\vec{\xi}$ and $(\vec{\omega} \times \vec{\theta})$, and φ = the angle between $\vec{\omega}$ and $\vec{\theta}$.

6). The parameter κ may be related with weak decay coupling constants, or equivalently with the Cabibbo-Kobayashi-Maskawa angles. Hence κ influences decay types.

9.3. Weak decay conditions. When a weak process is coupled with some external fields, energy exchange occurs. In general, gravity is much weaker than electromagnetic and strong interactions. Hence, ignoring the gravitational terms, the weak interacting field equations coupling external forces can be written as

$$(9.15) \quad \begin{aligned} & \partial^\nu W_{\nu\mu}^a - \frac{g_w}{2\hbar c} \varepsilon^{abc} g^{\nu\nu} W_{\nu\mu}^b W_\nu^c - g_w J_\mu^a \\ &= \frac{g_w}{\sqrt{\hbar c}} \zeta^a \left[\partial_\mu + \frac{e}{\hbar c} A_\mu + \frac{g_w \alpha_w^b}{\hbar c} W_\mu^b + \frac{g_s \alpha_s^k}{\hbar c} S_\mu^k + \frac{k_H^2}{4} x_\mu \right] \phi, \end{aligned}$$

where $\{\alpha_k^s\} = (\alpha_1^s, \dots, \alpha_8^s)$ is the $SU(3)$ tensor, S_μ^k ($1 \leq k \leq 8$) is the gauge potential of strong interaction, e is the electric charge whose sign is undetermined, and g_s is the strong charge.

As W_μ^a are the weak potential in the interior of a particle, ϕ is given by (9.12), and

$$(9.16) \quad W_\mu^a = 0 \quad \text{at} \quad r > r_0 = \frac{1}{k_H} \cong 10^{-16} \text{ cm}.$$

Take the gauge

$$(9.17) \quad \partial^\mu W_\mu^a = \text{const}.$$

and assume that W_μ^a are independent of t . Equations (9.15) are rewritten as

$$(9.18) \quad -\nabla^2 W_\mu^a - \frac{g_w^2}{(\hbar c)^{3/2}} \zeta^a \alpha_w^b \phi = g_w J_\mu^a + \frac{g_w}{2\hbar c} \varepsilon^{abc} g^{\nu\nu} W_{\nu\mu}^b W_\nu^c \\ + \frac{g_w}{\sqrt{\hbar c}} \zeta^a \left[\frac{e}{\hbar c} A_\mu + \frac{g_s \alpha_s^k}{\hbar c} S_\mu^k + \frac{k_H^2}{4} x_\mu \right] \phi + \frac{g_w}{\sqrt{\hbar c}} \zeta^a \partial_\mu \phi.$$

Multiplying both sides of (9.18) by W_μ^a and integrating the sum in $y \in \mathbb{R}^3$ with $|y| < r_0 = k_0^{-1}$, by (9.7), (9.16) and (9.17) we deduce that

$$(9.19) \quad \int_{B_{r_0}} |\nabla W|^2 dy - \int_{B_{r_0}} \frac{g_w^2}{(\hbar c)^{3/2}} \zeta^a \alpha_w^b W_\mu^b \cdot W_\mu^a \phi dy \\ = g_w \alpha_w^a \omega^a + \frac{g_w}{2\hbar c} \int_{B_{r_0}} \varepsilon^{abc} g^{\nu\nu} W_\mu^a W_{\nu\mu}^b W_\nu^c dy \\ + \int_{B_{r_0}} \frac{g_w}{\sqrt{\hbar c}} \left[\frac{e}{\hbar c} A_\mu + \frac{g_s \alpha_s^k}{\hbar c} S_\mu^k + \frac{k_H^2}{4} x_\mu \right] W_\mu^a \zeta^a \phi dy,$$

where α_w^a and ω^a are as in (9.7) and (9.9), ϕ as (9.12), and $B_{r_0} = \{y \in \mathbb{R}^3 \mid |y| < r_0\}$.

Approximatively, taking the spheric coordinates we have

$$(9.20) \quad A = \int_{B_{r_0}} \frac{e^{-k_0 r}}{r} A_\mu W_\mu^a \zeta^a dy = \frac{r_0^2}{2} |\Omega| \mathcal{A}_\mu \mathcal{W}_\mu,$$

$$(9.21) \quad S = \int_{B_{r_0}} \frac{e^{-k_0 r}}{r} S_\mu W_\mu^a \zeta^a dy = \frac{r_0^2}{2} |\Omega| S_\mu \mathcal{W}_\mu,$$

where $|\Omega|$ is the area of the unit sphere, and

$$\mathcal{A}_\mu = A_\mu(0), \quad \mathcal{S}_\mu = \alpha_s^k S_\mu^k(0), \quad \mathcal{W}_\mu = W_\mu^a(0) \zeta^a.$$

Let

$$(9.22) \quad M = \int_{B_{r_0}} |\nabla W|^2 dy, \quad V = \int_{B_{r_0}} \frac{g_w^2}{(\hbar c)^{3/2}} \zeta^a \alpha_w^b W_\mu^b \cdot W_\mu^a \phi dy, \\ I = \int_{B_{r_0}} \varepsilon^{abc} g^{\nu\nu} W_\mu^a W_{\nu\mu}^b W_\nu^c dy, \quad \Phi = \alpha_w^a \omega^a, \\ H = \frac{1}{4} \int_{B_{r_0}} \frac{g_w}{\sqrt{\hbar c}} x_\mu W_\mu^a \zeta^a \phi dy.$$

Then (9.18) is rewritten as

$$(9.23) \quad M - V = g_w \Phi + \frac{g_w}{2\hbar c} I + \frac{\kappa e g_w^2}{\hbar c} A + \frac{\kappa g_s g^2}{\hbar c} S + \kappa g_w^2 H.$$

Therefore, based on the criterion (9.13) and (9.14), we derive from (9.23) that for a particle under an external electromagnetic and strong fields A_μ and S_μ^k , the condition that it can decay is

$$(9.24) \quad \left[\Phi + \frac{I}{2} \right] + \kappa g \left[\frac{e}{\hbar c} A + \frac{g_s}{\hbar c} S + H \right] \geq 0.$$

By (9.20)-(9.22), the first part in the right-hand side of (9.24) represents the weak field energy generated by the weak charge g_w , and the second part is the energy generated by external fields.

9.4. Weak interaction potential. By (9.2), the time-components W_0^a of W_μ^a ($a = 1, 2, 3$) represent the weak charge potentials generated by the weak charge g_w . We now derive an approximate formula for the total potential:

$$(9.25) \quad W = \alpha_w^a W_0^a, \quad \text{where } |\alpha_w|^2 = \alpha_w^a \alpha_w^a = 1.$$

Assuming that W_μ^a are independent of time and taking linear approximation, from (5.27) we have

$$(9.26) \quad -\nabla^2 W = g_w \alpha_w^a J_0^a + \frac{g_w}{\sqrt{\hbar c}} \alpha_w^a \zeta^a \left(\frac{k_0^2}{4} c\tau - \frac{g_w}{\hbar c} W \right) \phi,$$

where τ is the lifetime of the Higgs, and

$$(9.27) \quad \phi = \theta + \phi_0,$$

where ϕ_0 is given by (9.12) and θ is a constant. Taking a translation

$$W \longrightarrow W + \frac{k_0^2 \hbar c^2 \tau}{4g_w},$$

and by $J_0^a = \alpha_w^a \delta(x)$, equations (9.26) and (9.27) become

$$(9.28) \quad -\nabla^2 W + k_1^2 W = g_w \delta(x) - \frac{g_w^2}{(\hbar c)^{3/2}} K W \phi_0,$$

where

$$(9.29) \quad k_1^2 = \frac{g_w^2}{(\hbar c)^{3/2}} \alpha_w^a \zeta^a \theta, \quad K = \alpha_w^a \zeta^a.$$

Solutions of (9.28) and (9.29) can be expressed as

$$(9.30) \quad W = W_0 + W_1 + \cdots,$$

where W_n satisfy

$$(9.31) \quad -\nabla^2 W_0 + k_1^2 W_0 = g_w \delta(x),$$

$$(9.32) \quad -\nabla^2 W_n + k_1^2 W_n = -\left(\frac{g_w^2}{\hbar c}\right)^{3/2} \frac{A}{r} e^{-k_1 r} W_{n-1} \quad \text{for } n = 1, 2, \cdots,$$

and $A = K\kappa$. The solution of (9.31) is

$$(9.33) \quad W_0 = \frac{g_w}{r} e^{-k_1 r}.$$

When $n = 1$, (9.32) is given by

$$(9.34) \quad \nabla^2 W_1 - k_1^2 W_1 = A \left(\frac{g_w^2}{\hbar c}\right)^{3/2} \frac{g_w}{r^2} e^{-(k_0+k_1)r}.$$

Let W_1 be radial symmetric and in the form

$$(9.35) \quad W_1 = A \left(\frac{g_w^2}{\hbar c}\right)^{3/2} g_w e^{-(k_0+k_1)r} \varphi_1(r).$$

Then (9.34) implies that

$$(9.36) \quad \varphi_1'' + 2 \left(\frac{1}{r} - K_1 \right) \varphi_1' - \frac{2K_1}{r} \varphi_1 + (K_1^2 - k_1^2) \varphi_1 = \frac{1}{r^2},$$

where $K_1 = k_0 + k_1$.

Let φ_1 be expanded as

$$(9.37) \quad \varphi_1 = \sum_{k=0}^{\infty} p_k r^k \ln r + \sum_{k=0}^{\infty} q_k r^k.$$

Inserting (9.37) into (9.36) and comparing coefficients, we deduce that

$$\begin{aligned} p_0 &= 1, & p_1 &= \frac{2}{3}(1 + q_0), & p_2 &= \frac{5}{18}K_1^2 + \frac{1}{6}k_1^2, & \dots, \\ q_0 \text{ and } q_1 &\text{ are free,} & q_2 &= -\frac{1}{36}K_1^2 - \frac{5}{12} + K_1\beta_1, & \dots. \end{aligned}$$

Then we infer from (9.32) that

$$(9.38) \quad W_n = A^n \left(\frac{g_w^2}{\hbar c} \right)^{3n/2} g_w e^{-K_n r} \varphi_1(r), \quad K_n = k_0 + nk_1 \quad \text{for } n \geq 1,$$

and φ_n satisfies

$$(9.39) \quad \varphi_n'' + 2 \left(\frac{1}{r} - K_n \right) \varphi_n' - \frac{2K_n}{r} \varphi_n + (K_n^2 - k_1^2) \varphi_n = \frac{\varphi_{n-1}}{r}.$$

The solution of this equation is in the form

$$(9.40) \quad \varphi_n = \sum_{k=n-1}^{\infty} p_k^n r^k \ln r + \sum_{k=n-1}^{\infty} q_k^n r^k,$$

where p_k^n and q_k^n depend on the free parameters q_0 and q_1 .

Hence by (9.30) and (9.38), the solution W of (9.28) can be expressed as

$$(9.41) \quad W = g_w e^{-k_1 r} \left[\frac{1}{r} - \sum_{n=1}^{\infty} A^n \left(\frac{g_w^2}{\hbar c} \right)^{3n/2} e^{-(k_0 + (n-1)k_1)r} \psi_n \right],$$

where $\psi_n = -\varphi_n$ and φ_n is given by (9.40).

The function $\Psi = e^{-k_0 r} \psi$ with

$$\psi = \sum_{n=1}^{\infty} A^n \left(\frac{g_w^2}{\hbar c} \right)^{3n/2} g_w e^{-nk_1 r} \psi_n$$

is the solution of

$$(9.42) \quad \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \Psi - k_1^2 \Psi = - \left(\frac{g_w^2}{\hbar c} \right)^{3/2} \frac{A}{r} e^{-k_1 r} \Psi - \left(\frac{g_w^2}{\hbar c} \right)^{3/2} \frac{A g_w}{r^2} e^{-(k_0 + k_1)r}.$$

Now we supply ψ with the following initial conditions:

$$(9.43) \quad \psi(r_1) = a_0, \quad \psi'(r_1) = a_1 \quad \text{at } r_1 > 0.$$

In summary, we have derived the weak potential and weak force formula given by

$$(9.44) \quad W = g_w e^{-k_1 r} \left[\frac{1}{r} - e^{-k_0 r} \psi(r) \right],$$

$$(9.45) \quad F = g_w^2 e^{-k_1 r} \left[\frac{k_1}{r} + \frac{1}{r^2} - (K_1 \psi - \psi') e^{-k_0 r} \right],$$

where $K_1 = k_0 + k_1$, $k_0 = m_H c/\hbar$, $k_1 = m_W c/\hbar$, m_H and m_W are the masses of the Higgs and W^\pm or Z bosons, and by (9.43), $\psi(r)$ can be approximately written as

$$(9.46) \quad \psi(r) = a_0 + a_1 r \quad \text{near } r = r_1 = \frac{1}{k_0}.$$

Thus the weak force becomes

$$(9.47) \quad F = g_w^2 e^{-k_1 r} \left[\frac{k_1}{r} + \frac{1}{r^2} - (a_0 - a_1 + a_1 r) e^{-k_0 r} \right].$$

Based on known physical facts, we have

$$(9.48) \quad a_0 > 0, \quad a_1 \leq 0, \quad a_0 - a_1 \gg k_1^2.$$

Hence we have derived the following physical conclusions:

- (1) A particle with weak charge g_w will generate a weak force F exerted on another with weak charge g_w , and the force F is given by (9.45) or (9.47).
- (2) By (9.41 and (9.45), there is a radius $r_0 > 0$ such that F is repelling for $r < r_0$, and

$$F \rightarrow \infty \quad \text{as } r \rightarrow 0.$$

- (3) By (9.47) and (9.48), F has an attractive region: $r_0 < r < r_1$.
- (4) The weak interaction force F is of short-range:

$$F \sim 0 \quad \text{for } r > \frac{1}{k_1} \sim 10^{-16} \text{ cm}.$$

10. CONSISTENCY WITH GWS ELECTROWEAK THEORY

The main objective of this section is to study the consistency of the new electroweak theory based on PID and PRI with the classical GWS electroweak theory.

10.1. GWS action. For comparison, we first introduce the classical Glashow-Weinberg-Salam electroweak theory, which is a $U(1) \otimes SU(2)$ gauge theory. We adopt here the classical notations. The action is given by

$$(10.1) \quad L_{GWS} = \int [\mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_H] dx.$$

Here \mathcal{L}_G is the gauge part, \mathcal{L}_F is the fermionic part, and \mathcal{L}_H is the Higgs sector:

$$(10.2) \quad \begin{aligned} \mathcal{L}_G &= -\frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \\ \mathcal{L}_F &= i \bar{L} \gamma^\mu D_\mu L + i \bar{e}^R \gamma^\mu D_\mu e^R, \\ \mathcal{L}_H &= D_\mu \phi^\dagger D^\mu \phi + \lambda (\phi^\dagger \phi - a^2)^2 + G_e (\bar{L} \phi e^R + \bar{e}^R \phi^\dagger L), \end{aligned}$$

where G_e and $a > 0$ are constants, $L = (\nu_e, e^L)$, e^R is the wave function of right-hand electron, ϕ is the Higgs scalar field, and

$$\begin{aligned} W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g_1 \epsilon^{abc} W_\mu^b W_\nu^c, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ D_\mu e^R &= (\partial_\mu + i g_2 B_\mu) e^R, \\ D_\mu L &= (\partial_\mu + i \frac{g_2}{2} B_\mu - i \frac{g_1}{2} W_\mu^a \sigma_a) L, \\ D_\mu \phi &= (\partial_\mu - i \frac{g_2}{2} B_\mu - i \frac{g_1}{2} W_\mu^a \sigma_a) \phi, \end{aligned}$$

Here g_1 and g_2 are coupling constants, ε^{kij} ($1 \leq k, i, j \leq 3$) are the structural constants of $SU(2)$, σ_k ($1 \leq k \leq 3$) are the Pauli matrices, $\{W_\mu^a\}$ is the Yang-Mills gauge field corresponding to the k -th generator of $SU(2)$, and $\{B_\mu\}$ is the gauge field with respect to $U(1)$.

We note that B_μ does not represent the electromagnetic potential A_μ , and the Higgs field ϕ is a complex doublet given by

$$\phi = (\phi^+, \phi^0)^T,$$

which has charge (1,0).

The action (10.1) is invariant under the $SU(2)$ gauge transformation

$$\begin{aligned} L &\rightarrow e^{\frac{i}{2}\theta^a\sigma_a}L, \\ \phi &\rightarrow e^{-\frac{i}{2}\theta^a\sigma_a}\phi, \\ e^R &\rightarrow e^R, \\ W_\mu^a &\rightarrow W_\mu^a - \frac{2}{g_1}\partial_\mu\theta^a + \varepsilon^{abc}\theta^bW_\mu^c, \end{aligned} \quad (10.3)$$

and the $U(1)$ gauge transformation

$$\begin{aligned} L &\rightarrow e^{\frac{i}{2}\beta}L, \\ \phi &\rightarrow e^{-\frac{i}{2}\beta}\phi, \\ e^R &\rightarrow e^{i\beta}e^R, \\ W_\mu^a &\rightarrow W_\mu^a - \frac{2}{g_2}\partial_\mu\beta, \\ B_\mu &\rightarrow B_\mu + \frac{2}{g_2}\partial_\mu\beta. \end{aligned} \quad (10.4)$$

We notice from (10.2) and (10.3) that \mathcal{L}_F contains the following terms:

$$(10.5) \quad W_\mu^a J_\mu^a, \quad J_\mu^a = \bar{L}\gamma_\mu\sigma_a L.$$

These terms are crucial in the weak interaction theory because under a unitary transformation

$$(10.6) \quad \begin{pmatrix} \sigma^+ \\ \sigma^- \\ \sigma^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix},$$

these terms become

$$\begin{aligned} (10.7) \quad W_\mu^{\mu\pm} J_\mu^\pm, \quad W_\mu^{\mu 3} J_\mu^3, \quad W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2), \\ J_\mu^{\mu\pm} = \bar{L}\gamma^\mu\sigma_\pm L, \quad J_\mu^3 = \bar{L}\gamma^\mu\sigma_3 L, \end{aligned}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, and J_μ^\pm are the charged currents consistent with the classical V-A theory and the intermediate vector boson (IVB) theory, which are two successful models at low energies.

Physically, W_μ^\pm particles are vector intermediate bosons having mass m_W , which should satisfy the Klein-Gordon equation

$$\partial^\mu\partial_\mu W_\nu^\pm + k^2 W_\nu^\pm = o(W^\pm),$$

where $o(W^\pm)$ stands for the higher order terms of W^\pm , and $k = m_W c/\hbar$. However, we find that the variational equations of the action (10.1) have the form

$$\frac{\delta L_{GWS}}{\delta W_\mu^a} = \partial^\mu W_{\mu\nu}^a + o(W) = 0,$$

which implies that W_μ^\pm in (10.7) would be massless, contradicting with the fact that W^\pm are massive.

Higgs mechanism provides a resolution. We see from (10.2) and (10.3) that $\phi_0 = (0, a)^T$ is an extremum point of (10.1), i.e. for $\Phi = (W, B, L, R, \phi)$, $\Phi_0 = (0, 0, 0, 0, \phi_0)$ is a solution of

$$\delta L_{GWS} = 0.$$

Consider the translation

$$(10.8) \quad \Phi = \Phi' + \Phi_0, \quad \Phi' = (W', B', L', R', \phi').$$

Then the variational equations of L_{GWS} for Φ' are given by (for simplicity, omitting the primes)

$$(10.9) \quad \left(\begin{array}{c} \frac{\delta}{\delta W} L_{GWS} \\ \frac{\delta}{\delta B} L_{GWS} \end{array} \right) = \left(\begin{array}{c} \partial^\mu W_{\mu\nu} \\ \partial^\mu B_{\mu\nu} \end{array} \right) + M \left(\begin{array}{c} W_\nu \\ B_\nu \end{array} \right) + o(W, B) = 0,$$

where M is the mass matrix induced by Φ_0 . It is clear that (10.9) is no longer covariant, or equivalently L_{GWS} breaks the symmetry for Φ' for (10.3)-(10.3). But the particles described by (W', B') receive masses due to the symmetry breaking.

10.2. Weinberg-Salam electroweak theory. We now recapitulate the WS electroweak theory. In (10.1)-(10.3), replace $\phi = (\phi^+, \phi^0)^T$ by $\phi = (0, \varphi)$, then the system is simplified and still invariant under the transformations (10.3) and (10.4), and avoids the difficulty that there exists a charged and massless bosonic field ϕ^+ in the classical GWS model. In this case, the Higgs action in (10.2) becomes

$$(10.10) \quad \mathcal{L}_H = \partial^\mu \varphi \partial_\mu \varphi + \varphi^2 \left[\frac{g_1^2}{4} W_\mu^a W^{\mu a} + \frac{g_2^2}{4} B_\mu B^\mu - \frac{g_1 g_2}{2} B^\mu W_\mu^3 \right] - \lambda(\varphi^2 - a^2)^2 + G_e \varphi (\bar{e}^L e^R + \bar{e}^R e^L).$$

Then the Euler-Lagrange equations of the action (10.1) are given by

$$(10.11) \quad \partial^\nu W_{\nu\mu}^1 - \frac{g_1}{2} g^{\nu\nu} (W_{\nu\mu}^2 W_\mu^3 - W_{\nu\mu}^3 W_\mu^2) + \frac{g_1}{2} J_\mu^1 + \frac{g_1^2}{2} \varphi^2 W_\mu^1 = 0,$$

$$(10.12) \quad \partial^\nu W_{\nu\mu}^2 - \frac{g_1}{2} g^{\nu\nu} (W_{\nu\mu}^3 W_\mu^1 - W_{\nu\mu}^1 W_\mu^3) + \frac{g_1}{2} J_\mu^2 + \frac{g_1^2}{2} \varphi^2 W_\mu^2 = 0,$$

$$(10.13) \quad \partial^\nu W_{\nu\mu}^3 - \frac{g_1}{2} g^{\nu\nu} (W_{\nu\mu}^1 W_\mu^2 - W_{\nu\mu}^2 W_\mu^1) + \frac{g_1}{2} J_\mu^3 + \frac{g_1}{2} \varphi^2 (g_1 w_\mu^3 - g_2 B_\mu) = 0,$$

$$(10.14) \quad \partial^\nu B_{\nu\mu} - \frac{g_2}{2} J_\mu^L - g_2 J_\mu^R + \frac{g_2}{2} \varphi^2 (g_2 B_\mu - g_1 W_\mu^3) = 0,$$

$$(10.15) \quad i\gamma^\mu (\partial_\mu + i\frac{g_2}{2} B_\mu - i\frac{g_1}{2} W_\mu^a \sigma_a) \begin{pmatrix} \nu \\ e^L \end{pmatrix} + G_e e^R \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = 0,$$

$$(10.16) \quad i\gamma^\mu (\partial_\mu + ig_2 B_\mu) e^R + G_e \varphi e^L = 0,$$

$$(10.17) \quad \partial^\mu \partial_\mu \varphi - \frac{1}{2} \varphi (g_1^2 W_\mu^a W^{\mu a} + g_2^2 B_\mu B^\mu - 2g_1 g_2 W_\mu^3 B^\mu) + 4\lambda a^2 \varphi - 4\lambda \varphi^3 - G_e (\bar{e}^L e^R + \bar{e}^R e^L) = 0.$$

Under the translation (10.8), or equivalently inserting

$$(10.18) \quad \varphi = \phi_0 + a$$

into (10.11)-(10.16) we obtain

$$(10.19) \quad \partial^\nu \widetilde{W}_{\nu\mu}^1 + \frac{g_1^2 a^2}{2} W_\mu^1 + \frac{g_1}{2} J_\mu^1 = o(W, \phi_0),$$

$$(10.20) \quad \partial^\nu \widetilde{W}_{\nu\mu}^2 + \frac{g_1^2 a^2}{2} W_\mu^2 + \frac{g_1}{2} J_\mu^2 = o(W, \phi_0),$$

$$(10.21) \quad \partial^\nu \widetilde{W}_{\nu\mu}^3 + \frac{g_1^2 a^2}{2} W_\mu^3 - \frac{g_1 g_2 a^2}{2} B_\mu + \frac{g_1}{2} J_\mu^3 = o(W, B, \phi_0),$$

$$(10.22) \quad \partial^\nu B_{\nu\mu} + \frac{g_2^2 a^2}{2} B_\mu - \frac{g_1 g_2 a^2}{2} W_\mu^3 - g_2 J_\mu^R - \frac{g_2}{2} J_\mu^L = o(W, B, \phi_0),$$

$$(10.23) \quad i\gamma^\mu D_\mu \nu_e = 0,$$

$$(10.24) \quad i\gamma^\mu D_\mu e^L + a G_e e^R + G_e \phi_0 e^R = 0,$$

$$(10.25) \quad i\gamma^\mu D_\mu e^R + a G_e e^L + G_e \phi_0 e^L = 0,$$

where

$$(10.26) \quad \widetilde{W}_{\nu\mu}^a = \partial_\nu W_\mu^a - \partial_\mu W_\nu^a, \quad B_{\nu\mu} = \partial_\nu B_\mu - \partial_\mu B_\nu.$$

From (10.19)-(10.22) we can find the mass terms

$$(10.27) \quad M \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \frac{g_1^2 a^2}{2} & 0 & 0 & 0 \\ 0 & \frac{g_1^2 a^2}{2} & 0 & 0 \\ 0 & 0 & \frac{g_1^2 a^2}{2} & -\frac{g_1 g_2 a^2}{2} \\ 0 & 0 & -\frac{g_1 g_2 a^2}{2} & \frac{g_2^2 a^2}{2} \end{pmatrix} \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix}$$

and the current terms

$$(10.28) \quad J_\mu = \left(\frac{g_1}{2} J_\mu^1, \quad \frac{g_1}{2} J_\mu^2, \quad \frac{g_1}{2} J_\mu^3, \quad -g_2 J_\mu^R - \frac{g_1}{2} J_\mu^L \right)^T.$$

In order to generate masses, we have to diagonalize the matrix of (10.27), by a rotating transformation for (W_μ^3, B_μ) as

$$(10.29) \quad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \frac{g_1}{|g|} & \frac{g_2}{|g|} \\ -\frac{g_2}{|g|} & \frac{g_1}{|g|} \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix},$$

where $|g| = (g_1^2 + g_2^2)^{1/2}$. On the other hand, the charged currents J_μ^\pm are given by (10.7) which are derived by the transformation (10.6), or equivalently by the unitary rotation of (W_μ^1, W_μ^2) as

$$(10.30) \quad \begin{pmatrix} W_\mu^+ \\ W_\mu^- \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \end{pmatrix}.$$

Hence, under the following unitary transformation

$$(10.31) \quad \begin{pmatrix} W_\mu^+ \\ W_\mu^- \\ Z_\mu \\ A_\mu \end{pmatrix} = U \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix}, \quad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{g_1}{|g|} & \frac{g_2}{|g|} \\ 0 & 0 & -\frac{g_2}{|g|} & \frac{g_1}{|g|} \end{pmatrix},$$

the mass matrix M in (10.27) becomes

$$(10.32) \quad U M U^\dagger = \frac{c^2}{\hbar^2} \begin{pmatrix} m_W^2 & 0 & 0 & 0 \\ 0 & m_W^2 & 0 & 0 \\ 0 & 0 & m_Z^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the current J_μ in (10.28) is as

$$(10.33) \quad \left(\frac{g_1}{\sqrt{2}} J_\mu^+, \quad \frac{g_1}{\sqrt{2}} J_\mu^-, \quad |g| J_\mu^{NC}, \quad e J_\mu^{em} \right)^T = U J_\mu.$$

Also, equations (10.19)-(10.22) become

$$(10.34) \quad \begin{aligned} \partial^\nu (\partial_\nu W_\mu^+ - \partial_\mu W_\nu^+) + \left(\frac{cm_W}{\hbar} \right)^2 W_\mu^+ + \frac{g_1}{\sqrt{2}} J_\mu^+ &= o(\Phi), \\ \partial^\nu (\partial_\nu W_\mu^- - \partial_\mu W_\nu^-) + \left(\frac{cm_W}{\hbar} \right)^2 W_\mu^- + \frac{g_1}{\sqrt{2}} J_\mu^- &= o(\Phi), \\ \partial^\nu (\partial_\nu Z_\mu - \partial_\mu Z_\nu) + \left(\frac{cm_Z}{\hbar} \right)^2 Z_\mu + |g| J_\mu^{NC} &= o(\Phi), \\ \partial^\nu (\partial_\nu A_\mu - \partial_\mu A_\nu) - e J_\mu^{em} &= o(\Phi), \end{aligned}$$

where $\Phi = (W^+, W^-, Z, A, \phi_0)$.

The above field equations (10.31)-(10.34) lead to the following physical conclusions as part of the classical electroweak theory:

1). When the Higgs field φ possesses a nonzero vacuum state as (10.18), the gauge symmetry breaks, and the fields W_μ^a and B_μ are recombined to yield a changed doublet of massive vector intermediate bosons W_μ^\pm , a neutral massive vector boson Z_μ , and a massless photon field A_μ :

$$(10.35) \quad \begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}} (W_\mu^1 \pm W_\mu^2), \\ Z_\mu &= \frac{1}{|g|} (g_1 W_\mu^3 + g_2 B_\mu) = \cos \theta_W W_\mu^3 + \sin \theta_W B_\mu, \\ A_\mu &= \frac{1}{|g|} (-g_2 W_\mu^3 + g_1 B_\mu) = -\sin \theta_W W_\mu^3 + \cos \theta_W B_\mu, \end{aligned}$$

where θ_W is the Weinberg angle defined as

$$\cos \theta_W = \frac{g_1}{|g|} = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \quad \sin \theta_W = \frac{g_2}{|g|} = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}.$$

2). The masses of W_μ^\pm and Z are as in (10.32) given by

$$(10.36) \quad m_{W^+} = m_{W^-} = \frac{a\hbar}{\sqrt{2}c} g_1, \quad m_Z = \frac{a\hbar}{\sqrt{2}c} |g|$$

3). The electric charge in (10.34) is

$$(10.37) \quad e = g_1 \sin \theta_W.$$

4). Both charged currents J_μ^\pm and the neutral current J_μ^{NC} appearing in (10.34) are derived from (10.28) and (10.33), given by

$$(10.38) \quad J_\mu^\pm = \frac{1}{2} (J_\mu^1 \pm i J_\mu^2), \quad J_\mu^{NC} = \frac{1}{2} [\cos^2 \theta_W J_\mu^3 - \sin^2 \theta_W J_\mu^L - 2 \sin^2 \theta_W J_\mu^R],$$

where J_μ^a is as in (10.5), and

$$\begin{aligned} J_\mu^3 &= \bar{\nu}_e \gamma_\mu \nu_e - \bar{e}^L \gamma_\mu e^L, & J_\mu^R &= \bar{e}^R \gamma_\mu e^R, \\ J_\mu^L &= \bar{\nu}_e \gamma_\mu \nu_e + \bar{e}^L \gamma_\mu e^L, & J_\mu^{em} &= \frac{1}{2} J_\mu^3 + \frac{1}{2} J_\mu^L + J_\mu^R = \bar{\nu}_e \gamma_\mu \nu_e + \bar{e}^R \gamma_\mu e^R. \end{aligned}$$

Here J_μ^{em} is the electric current.

5). By (10.24) and (10.25) the mass of an electron is given by

$$(10.39) \quad m_e = a G_e.$$

6). Finally, (10.17) is the Higgs field equation, from which we derive the mass of the Higgs particle as

$$(10.40) \quad m_H = 2a\sqrt{\lambda}.$$

Two remarks are now in order.

Remark 10.1. From (10.39) and (10.40) we see that the electron mass m_e and the Higgs particle mass m_H can not be determined by the electroweak theory, and their values are also helpless for determining the masses of W_μ^\pm and Z_μ . By the $V - A$ theory, we have

$$\frac{G_F}{\sqrt{2}} = \frac{g_1^2}{8m_W^2} = \frac{1}{4a^2},$$

where G_F is the Fermi constant. Therefore,

$$(10.41) \quad a^2 = \frac{1}{2\sqrt{2}G_F}.$$

Experimentally the value of θ_W is determined by

$$\sin^2 \theta_W = 0.2325 \pm 0.008.$$

Then, from (10.36), (10.37) and (10.41) it follows that

$$\begin{aligned} m_W &= 80.22 \pm 0.26 \text{ GeV}/c^2, \\ m_Z &= 91.173 \pm 0.020 \text{ GeV}/c^2. \end{aligned}$$

Remark 10.2. It is worth mentioning that the transformation (10.31) corresponds to a mixed transformation of $U(1)$ generator $\sigma_0 = 1$ and $SU(2)$ generators σ_a :

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_0 \end{pmatrix} = U \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_0 \end{pmatrix}.$$

Consequently, the GWS electroweak model cannot be decoupled to study individual interactions. In other words, the classical electroweak theory violates the principle of representation invariance (PRI).

10.3. Electroweak theory obeying PRI. We have seen that the classical electroweak theory violates PRI. In the following we develop a much simpler electroweak theory based on PID, which not only satisfies the PRI, but also leads to the same outcomes as the classical electroweak theory.

According to the IVB theory of weak interactions, the charged currents and the neutral currents are

$$(10.42) \quad j_\mu^\pm = \frac{g_1}{\sqrt{2}} J_\mu^\pm, \quad j_\mu^{NC} = \frac{g_1}{\cos \theta_W} J_\mu^{NC}.$$

In particular, we see that the WS theory gives rise to the ratio

$$j_\mu^\pm / j_\mu^{NC} = \frac{1}{\sqrt{2}} \cos \theta_W J_\mu^\pm / J_\mu^{NC},$$

which the new theory should retain as well.

We note that the doublets

$$(10.43) \quad \tilde{e}^R = \begin{pmatrix} e^R \\ 0 \end{pmatrix}, \quad \tilde{e}^L = \begin{pmatrix} 0 \\ e^L \end{pmatrix}, \quad \tilde{\nu}_e = \begin{pmatrix} \nu_e \\ 0 \end{pmatrix}$$

are $SU(2)$ symmetric, i.e. they can not be distinguished by themselves. Hence, the fermionic action \mathcal{L}_F in general is taken in the form

$$\begin{aligned} \mathcal{L}_F = & \bar{L}(i\gamma^\mu D_\mu - m^L)L + \alpha_1 \tilde{e}^L(i\gamma^\mu D_\mu - m^L)\tilde{e}^L \\ & + \alpha_2 \tilde{e}^R(i\gamma^\mu D_\mu - m^R)\tilde{e}^R + \alpha_3 i\tilde{\nu}_e \gamma^\mu D_\mu \tilde{\nu}_e, \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants, m_e is the electron mass,

$$m^L = \begin{pmatrix} 0 & 0 \\ 0 & m_e \end{pmatrix}, \quad m^R = \begin{pmatrix} m_e & 0 \\ 0 & 0 \end{pmatrix}$$

are 2nd-order tensors for the transformation

$$\begin{aligned} L & \rightarrow e^{i\frac{1}{2}\theta^a \sigma_a} L, \\ \tilde{e}^L & \rightarrow e^{i\frac{1}{2}\theta^a \sigma_a} \tilde{e}^L, \\ \tilde{e}^R & \rightarrow e^{i\frac{1}{2}\theta^a \sigma_a} \tilde{e}^R, \end{aligned}$$

and

$$\begin{aligned} D_\mu \tilde{e}^L & = (\partial_\mu + i\beta_1 B_\mu - i\frac{g_1}{2} W_\mu^a \sigma_a) \tilde{e}^L, \\ D_\mu \tilde{e}^R & = (\partial_\mu + i\beta_2 B_\mu - i\frac{g_1}{2} W_\mu^a \sigma_a) \tilde{e}^R, \\ D_\mu \tilde{\nu}_e & = (\partial_\mu + i\beta_3 B_\mu - i\frac{g_1}{2} W_\mu^a \sigma_a) \tilde{\nu}_e. \end{aligned}$$

By (10.43), \mathcal{L}_F can be equivalently written as

$$\begin{aligned} (10.44) \quad \mathcal{L}_F = & \bar{L}(i\gamma^\mu D_\mu - m^L)L + i\alpha_3 \bar{\nu}_e \gamma^\mu (\partial_\mu + i\beta_3 B_\mu - i\frac{g_1}{2} W_\mu^3) \nu_e \\ & - \alpha_1 \bar{e}^L \left[i\gamma^\mu (\partial_\mu + i\beta_1 B_\mu - i\frac{g_1}{2} W_\mu^3) - m_e \right] e^L \\ & + \alpha_2 \bar{e}^R \left[i\gamma^\mu (\partial_\mu + i\beta_2 B_\mu - i\frac{g_1}{2} W_\mu^3) - m_e \right] e^R. \end{aligned}$$

If we regard W_μ^3 as Z_μ , B_μ as A_μ and g_2 as e , then the currents derived from (10.44) are as follows

$$\begin{aligned} (10.45) \quad J_\mu^\pm & = \bar{L} \gamma_\mu \sigma^\pm L, \quad \sigma^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2), \\ J_\mu^{NC} & = \frac{\cos \theta_W}{2} [J_\mu^3 - \alpha_1 \bar{e}^L \gamma_\mu e^L + \alpha_2 \bar{e}^R \gamma_\mu e^R + \alpha_3 \bar{\nu}_e \gamma_\mu \nu_e], \\ J_\mu^{em} & = \frac{1}{2} [\bar{L} \gamma_\mu L - \frac{\alpha_1 \beta_1}{e} \bar{e}^L \gamma_\mu e^L + \frac{\alpha_2 \beta_2}{e} \bar{e}^R \gamma_\mu e^R + \frac{\alpha_3 \beta_3}{e} \bar{\nu}_e \gamma_\mu \nu_e]. \end{aligned}$$

The currents in (10.45) will be utterly the same as those from the classical electroweak theory if the parameters α_k and β_k ($1 \leq k \leq 3$) are chosen properly.

Now, we take the action

$$(10.46) \quad L = \int [\mathcal{L}_G + \mathcal{L}_F] dx,$$

where \mathcal{L}_F is (10.44) and \mathcal{L}_G as in (10.3) with $B_\mu = A_\mu$ being the electromagnetic potential. The the variational equations of (10.46) with the div_A -free constraint are given by

$$(10.47) \quad \begin{aligned} \partial^\nu W_{\nu\mu}^1 - \frac{g_1}{2} g^{\nu\nu} (W_{\nu\mu}^2 W_\nu^3 - W_{\nu\mu}^3 W_\nu^2) + \frac{g_1}{2} J_\mu^1 \\ = \left[\eta^1 \partial_\mu + \frac{\eta^1}{4} \left(\frac{m_{HC}}{\hbar} \right)^2 x_\mu - k^2 W_\mu^1 \right] \phi, \end{aligned}$$

$$(10.48) \quad \begin{aligned} \partial^\nu W_{\nu\mu}^2 - \frac{g_1}{2} g^{\nu\nu} (W_{\nu\mu}^3 W_\nu^1 - W_{\nu\mu}^1 W_\nu^3) + \frac{g_1}{2} J_\mu^2 \\ = \left[\eta^2 \partial_\mu + \frac{\eta^2}{4} \left(\frac{m_{HC}}{\hbar} \right)^2 x_\mu - k^2 W_\mu^2 \right] \phi, \end{aligned}$$

$$(10.49) \quad \begin{aligned} \partial^\nu W_{\nu\mu}^3 - \frac{g_1}{2} g^{\nu\nu} (W_{\nu\mu}^1 W_\nu^2 - W_{\nu\mu}^2 W_\nu^1) + \frac{g_1}{\cos \theta_W} J_\mu^{NC} \\ = \left[\eta^3 \partial_\mu + \frac{\eta^3}{4} \left(\frac{m_{HC}}{\hbar} \right)^2 x_\mu - k_0^2 W_\mu^3 \right] \phi, \end{aligned}$$

$$(10.50) \quad \partial^\nu (\partial_\nu A_\mu - \partial_\mu A_\nu) - e J_\mu^{em} = 0,$$

where $\eta = (\eta^1, \eta^2, \eta^3)$ is the $SU(2)$ gauge tensor, ϕ is a scalar field, $J_\mu^\pm = \frac{1}{2}(J_\mu^1 \pm iJ_\mu^2)$, J_μ^{NC} , J_μ^{em} are as in (10.45), and

$$(\varepsilon_{ab}) = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k_0^2 \end{pmatrix}$$

is a 2nd-order $SU(2)$ tensor with

$$(10.51) \quad k = \frac{m_W c}{\hbar}, \quad k_0 = \frac{m_Z c}{\hbar}.$$

It is clear that the equations (10.47)-(10.50) are covariant under transformations of representations of $U(1) \times SU(2)$. Namely, PRI holds true for this model.

Then by taking the divergence on both sides of (10.47)-(10.49) and making the inner product with $\eta = (\eta^1, \eta^2, \eta^3)$, we obtain the field equation for ϕ :

$$(10.52) \quad \begin{aligned} \partial^\mu \partial_\mu \phi + \left(\frac{m_{HC}}{\hbar} \right)^2 \phi = & \eta^b \varepsilon_{ab} W_\mu^a \partial^\mu \phi + \frac{g_1}{2} \eta^a \partial^\mu J_\mu^a \\ & + \frac{g_1}{2} \eta^a \varepsilon^{abc} g^{\nu\alpha} \partial^\mu (W_{\nu\mu}^b W_\alpha^c), \end{aligned}$$

which is the field equation describing the Higgs particle.

From (10.47)-(10.49) we see that when ϕ possesses a nonzero ground state, i.e.

$$\phi = 1 + \phi_0,$$

then the intermediate vector bosons W^\pm and Z with masses m_W and m_Z are generated. Furthermore we can easily derive all six conclusions 1)-6) in the last subsection based on the classical electroweak theory.

11. INTERACTION POTENTIALS

All four forces are described by their corresponding potentials, which obey the field equations (4.22)-(4.26). In [10] and previous sections of this article, we have derived force formulas for gravitational fields, strong interaction, and the weak interaction. These formulas offer unified theories for dark energy and dark matter, for quark confinement and asymptotic freedom, for the nuclear forces, for the van der Waals force, and for the nature of short-range properties of strong and weak interactions. In this section, we synthesize these formulas and their physical implications.

11.1. Charge and Rotation Potentials. Each interaction in nature has its source, which we call charge, generating the corresponding force:

gravitation:	mass charge m ,
electromagnetism:	electric charge e ,
weak interaction:	weak charge g_w ,
strong interaction:	strong charge g_s .

An interaction force is the negative gradient of the corresponding charge potential Φ :

$$(11.1) \quad F = -K\nabla\Phi \quad \text{with } K \text{ being the corresponding charge.}$$

The precise definitions of these charge potentials are given as follows:

GRAVITATION. The gravitational field is described by the Riemannian metric $\{g_{\mu\nu}\}$ of the four-dimensional space-time, representing the gravitational potential. In a center mass field, its charge potential Φ_G is the time-component g_{00} of $\{g_{\mu\nu}\}$ [1, 10]:

$$(11.2) \quad \Phi_G = -\frac{c^2}{2}(1 + g_{00}),$$

and the gravitational force is given by

$$(11.3) \quad F_G = -\frac{c^2 m}{2} \nabla g_{00}.$$

ELECTROMAGNETISM. For the electromagnetic potential $A_\mu = (A_0, A_1, A_2, A_3)$, the time-component A_0 represents its charge potential:

$$\Phi_E = A_0,$$

and the space vector $\vec{A} = (A_1, A_2, A_3)$ represents the magnetic potential. Consequently, we have

$$(11.4) \quad \begin{aligned} F_E &= -e\nabla\Phi_E && \text{the electric charge force,} \\ F_M &= \frac{1}{c} e \vec{v} \times \text{curl} \vec{A} && \text{the Lorentz force acting on } e. \end{aligned}$$

For the current $J_\mu = (J_0, J_1, J_2, J_3)$ given by (4.18), J_0 is the electric charge density, and $\vec{J} = (J_1, J_2, J_3)$ is the electric current density.

WEAK INTERACTION. The weak field is the $SU(2)$ gauge potentials $\{W_\mu^a \mid a = 1, 2, 3\}$. The total weak potential is defined by

$$(11.5) \quad W_\mu = \alpha_a^w W_\mu^a,$$

a gauge representation invariant scalar, i.e. obeying PRI. Here α_a^w is as defined in (4.31) representing the distribution vector of weak charge. In the same spirit as the electromagnetism, we define

$$(11.6) \quad \begin{aligned} W_0 & \quad \text{the weak charge potential,} \\ \vec{W} = (W_1, W_2, W_3) & \quad \text{the weak rotational potential,} \end{aligned}$$

and the corresponding weak force and weak rotational force are given by

$$(11.7) \quad \begin{aligned} F_{WE} &= -g_w \nabla W_0, \\ F_{WM} &= -g_w \varepsilon^{abc} \alpha_w^a \vec{J}^b \times \nabla \vec{W}^c, \end{aligned}$$

where ε^{abc} are the structural constants of $SU(2)$ with the Pauli representation, \vec{J}^a and \vec{W}^a are the space vectors of J_μ^a and W_μ^a , F_{WE} is the weak force acting on a particle with one weak charge g_w , and F_{WM} is the weak rotational force, a similar object of magnetic force. We note that both F_{WE} and F_{WM} are gauge group representation invariant, i.e. they obey PRI.

For the weak charge current J_μ^a , $\alpha_a^w J_0^a$ represents the weak charge density, and $\alpha_a^w \vec{J}^a$ stands for the weak current density.

STRONG INTERACTION. QCD fields are the $SU(3)$ gauge potentials $\{S_\mu^k \mid k = 1, \dots, 8\}$, representing the eight force carrier gluons. Thanks to PRI again, the total potential is

$$S_\mu = \alpha_k^s S_\mu^k, \quad \alpha_k^s \text{ is as in (4.31).}$$

The zeroth component S_0 represents the strong-charge potential, and the spatial component $\vec{S} = (S_1, S_2, S_3)$ represent strong-rotational potential:

$$(11.8) \quad \begin{aligned} S_0 & \quad \text{the strong charge potential,} \\ \vec{S} = (S_1, S_2, S_3) & \quad \text{the strong rotational potential.} \end{aligned}$$

and the forces

$$(11.9) \quad \begin{aligned} F_{SE} &= -g_s \nabla S_0, \\ F_{SM} &= g_s \lambda^{kij} \alpha_s^k \vec{Q}^i \times \text{curl} \vec{S}^j, \end{aligned}$$

represent the strong acting forces generated by the strong charge g_s and the strong charge current Q_μ^k . It is clear that both F_{SE} and F_{SM} obey PRI.

For the strong charge current Q_μ^k , $\alpha_k^s Q_0^k$ represents the strong charge density, and $\alpha_k^s \vec{Q}^k$ stands for the weak current density, where $\vec{Q}^k = (Q_1^k, Q_2^k, Q_3^k)$.

11.2. Gravitational force. The decoupling gravitational field equations for gravity from (4.34) are given by [10]:

$$(11.10) \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} - \nabla_\mu \nabla_\nu \varphi,$$

When we consider a spherically symmetric central gravitational field, the metric is in a diagonal form:

$$ds^2 = g_0 c^2 dt^2 + g_{11} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

with

$$(11.11) \quad \begin{aligned} g_{00} &= -e^u, & g_{11} &= e^v, & g_{22} &= r^2, & g_{33} &= r^2 \sin^2 \theta, \\ u &= u(r), & v &= v(r), & \varphi &= \varphi(r). \end{aligned}$$

It follows from (11.10) that u , v and φ satisfy

$$(11.12) \quad \begin{aligned} u'' + \frac{2u'}{r} + \frac{u'}{2}(u' - v') &= \varphi'' - \frac{1}{2} \left[u' + v' - \frac{4}{r} \right] \varphi', \\ u'' - \frac{2v'}{r} + \frac{u'}{2}(u' - v') &= -\varphi'' + \frac{1}{2} \left[u' + v' + \frac{4}{r} \right] \varphi', \\ u' - v' + \frac{2}{r}(1 - e^v) &= r \left[\varphi'' + \frac{1}{2}(u' - v')\varphi' \right], \end{aligned}$$

supplemented with the following initial conditions

$$(11.13) \quad u'(r_0) = u_0, \quad v(r_0) = v_0, \quad \varphi(r_0) = \varphi_0, \quad r_0 > 0.$$

By (11.3), (11.11) and (11.12) we obtain the following approximate gravitational force formula:

$$(11.14) \quad F_G = mMG \left[-\frac{1}{r^2} - \frac{k_0}{r} + k_1 r \right].$$

By (11.13) there are three free parameters in F_G . Therefore the two parameters k_0 and k_1 are free. The parameters k_0 and k_1 can be estimated using the Rubin rotational curves and the acceleration of the expanding galaxies:

$$k_0 = 4 \times 10^{-18} km^{-1}, \quad k_1 = 10^{-57} km^{-3}.$$

We emphasize here that the formula (11.14) is only a simple approximation for illustrating some features of both dark matter and dark energy.

11.3. Coulomb law. The decoupled electromagnetic field equations with duality from (4.35) are written as

$$(11.15) \quad \partial^\nu \partial_\nu A_\mu = eJ_\mu + \left(\partial_\mu - \frac{\alpha^E e}{\hbar c} A_\mu \right) \phi^E,$$

$$(11.16) \quad \partial^\mu A_\mu = 0,$$

and taking divergence on both sides of (11.15), by (11.16) and

$$\partial^\mu J_\mu = 0,$$

we deduce that

$$(11.17) \quad \partial^\mu \partial_\mu \phi^E - \frac{\alpha^E e}{\hbar c} A_\mu \partial^\mu \phi^E = 0.$$

Consider the static electric state:

$$\frac{\partial A_\mu}{\partial t} = 0, \quad \frac{\partial \phi^E}{\partial t} = 0.$$

We then infer from (11.15)-(11.17) and $J_0 = \delta(x)$ that

$$(11.18) \quad \begin{aligned} -\Delta^2 A_0 &= e\delta(x) - \frac{\alpha^E e}{\hbar c} A_0 \phi^E, \\ -\Delta^2 \phi^E &= \frac{\alpha^E e}{\hbar c} \vec{A} \cdot \nabla \phi^E, \\ -\Delta^2 \vec{A} &= \nabla \phi^E - \frac{\alpha^E e}{\hbar c} A_\mu \phi^E, \\ \text{div } \vec{A} &= 0. \end{aligned}$$

The radial symmetric solution of (11.18) is given by

$$(11.19) \quad A_0 = \frac{e}{r}, \quad \phi^E = 0,$$

which is the Coulomb potential. Hence the potential derived from the duality equations of electromagnetism is entirely the same as that derived from the classical Maxwell equations.

11.4. Strong interaction potential. By (4.17) and (4.18) we deduce that

$$\partial^\mu Q_\mu^k = -\frac{2g_s}{\hbar c} f^{kji} S_\mu^i Q^{\mu j}.$$

In a particle, we have

$$\partial^\mu Q_\mu^k = -\frac{2g_s}{\hbar c} f^{kji} S_0^i(0) \alpha_s^j \theta_0 \delta(x), \quad \frac{\partial \phi^s}{\partial t} = \frac{\partial S_\mu^k}{\partial t} = 0.$$

With a linear approximation, we derive from (5.35) and (5.36) that

$$(11.20) \quad -\nabla^2 S_0 = g_s \theta_0 \delta(r) + \frac{g_s \zeta^k \alpha_s^k}{4\sqrt{\hbar c}} k_0^2 c \tau \phi^s,$$

$$(11.21) \quad -\nabla^2 \phi^s + k^2 \phi^s = \frac{g_s \theta_0 \kappa}{\rho} \delta(x) - k^2 \vec{x} \cdot \nabla \phi^s,$$

where

$$k = \frac{mc}{\hbar}, \quad S_0^i(0) = \frac{1}{|B_\rho|} \int_{B_\rho} S_0^i dv = \frac{\xi^i}{\rho}, \quad \kappa = \frac{2}{\sqrt{\hbar c}} f^{ijk} \frac{\alpha_s^i \xi^j \zeta^k}{|\zeta|^2}.$$

Here ρ is the radius of the related particle, and m and τ are the mass and lifetime of ϕ^s particle.

Then we derive from (11.20) and (11.21) three levels of strong interaction potentials: the quark potential S_q , the nucleon potential S_n and the atom/molecule potential S_a :

$$(11.22) \quad S_q = g_s \left[\frac{1}{r} - \frac{B k_0^2}{\rho_0} e^{-k_0 r} \varphi(r) \right],$$

$$(11.23) \quad S_n = 3 \left(\frac{\rho_0}{\rho_1} \right)^3 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_1} e^{-k_1 r} \varphi(r) \right],$$

$$(11.24) \quad S_a = 3N \left(\frac{\rho_0}{\rho_1} \right)^3 \left(\frac{\rho_1}{\rho_2} \right)^3 g_s \left[\frac{1}{r} - \frac{B_n k_1^2}{\rho_2} e^{-k_1 r} \varphi(r) \right],$$

where $\varphi(r) \sim r/2$ is a polynomial, B, B_n are constants, $k_0 = mc/\hbar$, $k_1 = m_\pi c/\hbar$, m is mass of the strong interaction Higgs particle, m_π is the mass of the Yukawa meson, ρ_0 is the effective quark radius, ρ_1 is the radius of a nucleon/hadron, and ρ_2 is the radius of an atom/molecule.

11.5. Weak interaction potential. As the intermediate vector bosons W^\pm , Z and the Higgs boson ϕ^w are massive, the decoupled weak interaction field equations from (4.36) are given by

$$(11.25) \quad \begin{aligned} \partial^\nu W_{\nu\mu}^a - \frac{g_w}{2\hbar c} \varepsilon^{abc} g^{\alpha\beta} W_{\alpha\mu}^b W_\beta^c - g_w J_\mu^a \\ = \frac{g_w}{\sqrt{\hbar c}} \eta^a \left[\partial_\mu - \frac{g_w}{\hbar c} \alpha_b^w W_\mu^b + \frac{1}{4} \left(\frac{m_{H^c}}{\hbar} \right)^2 x_\nu \right] \phi^w. \end{aligned}$$

As in the case for strong interaction, we can derive the field equation for the weak interaction potential from (11.25):

$$(11.26) \quad -\Delta W = g_w \delta(x) + \frac{g_w^2}{\sqrt{\hbar c}} \alpha_w^a \eta^a \left[\frac{k_H^2}{4} c\tau - \frac{g_w}{\hbar c} W \right] \phi^w,$$

where $W = \alpha_w^a W_0^a$ is as in (11.6), $k_H = cm_H/\hbar$, m_H and τ are the mass and lifetime of the Higgs,

$$\phi^w = \beta + \phi_0, \quad \beta \text{ is a constant.}$$

Take a translation

$$W \longrightarrow W + \frac{k_H^2 \hbar c^2 \tau}{4g_w}.$$

Then (11.26) becomes

$$(11.27) \quad -\Delta W + k_1^2 W = g_w \delta(x) - \frac{g_w^2}{(\hbar c)^{3/2}} K W \phi_0,$$

where

$$k_1^2 = \frac{g_w^2}{(\hbar c)^{3/2}} \alpha_w^a \zeta^a \beta, \quad K = \alpha_w^a \eta^a.$$

From (11.25) we also obtain a linearized approximation for ϕ_0 :

$$(11.28) \quad -\nabla^2 \phi_0 + k_H^2 = -\sqrt{\hbar c} \xi^a \partial^\mu J_\mu^a,$$

where $\xi^a = \eta^a/|\eta|^2$, and we have supplemented a gauge equation for compensating the generation of ϕ :

$$\frac{g_w}{\hbar c} \alpha_a^w \partial^\mu W_\mu^b = k_H^2.$$

As in (11.22)-(11.24), we derive from (11.27) and (11.28) the weak interaction potential:

$$(11.29) \quad W = g_w e^{-k_1 r} \left[\frac{1}{r} - e^{-k_0 r} \psi(r) \right],$$

where

$$\begin{aligned} \psi &= \sum_{n=1}^{\infty} A^n \left(\frac{g_w^2}{\hbar c} \right)^{3n/2} g_w e^{-n k_1 r} \psi_n, \\ \psi_n(r) &= \ln r \sum_{k=n-1}^{\infty} a_k^n r^k + \sum_{k=n-1}^{\infty} b_k^n r^k \end{aligned}$$

$k_1 = m_W c/\hbar$, $k_H = m_H c/\hbar$, m_W is the mass of W^\pm or Z boson, and m_H is the mass of the Higgs.

12. ENERGY LEVELS OF ELEMENTARY PARTICLES

12.1. Energy levels of particles. Let G_μ represent the potentials for three interactions as follows:

$$\begin{aligned} G_\mu &= A_\mu && \text{for electromagnetic interaction,} \\ G_\mu &= W_\mu = \alpha_a^w W_\mu^a && \text{for weak interaction,} \\ G_\mu &= S_\mu = \alpha_k^s S_\mu^k && \text{for strong interaction.} \end{aligned}$$

Let $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^T$ be the wave function describing elementary particles such as baryons, leptons and quarks. Then as in gauge theories, the wave function Ψ satisfies the Dirac equations:

$$(12.1) \quad \left(i\hbar \frac{\partial}{\partial t} - gG_0 - mc^2 \right) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \hbar c (\vec{\sigma} \cdot \vec{P}) \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix},$$

$$(12.2) \quad \left(i\hbar \frac{\partial}{\partial t} - gG_0 + mc^2 \right) \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \hbar c (\vec{\sigma} \cdot \vec{P}) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},$$

where g is the corresponding charge,

$$\vec{\sigma} \cdot \vec{P} = \left(-i\partial_1 - \frac{g}{\hbar c} G_1 \right) \sigma_1 + \left(-i\partial_2 - \frac{g}{\hbar c} G_2 \right) \sigma_2 + \left(-i\partial_3 - \frac{g}{\hbar c} G_3 \right) \sigma_3,$$

and σ_i are the Pauli matrices.

If G_μ is independent of time, then Ψ can be written as

$$(12.3) \quad \Psi = e^{-i(E-mc^2)t/\hbar} \psi(x).$$

We then infer from (12.1) and (12.2) that

$$(12.4) \quad (E - gG_0 - 2mc^2) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \hbar c (\vec{\sigma} \cdot \vec{P}) \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix},$$

$$(12.5) \quad (E - gG_0) \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \hbar c (\vec{\sigma} \cdot \vec{P}) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Physically we have

$$E - gG_0 = \text{kinetic energy} + \text{mass}.$$

Hence we can approximately take

$$E - gG_0 = \varepsilon = \text{the average of kinetic energy} + \text{mass}.$$

Then (12.5) becomes

$$(12.6) \quad \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \hbar c (\varepsilon^{-1} \vec{\sigma} \cdot \vec{P}) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Inserting (12.6) into (12.4) leads to

$$(12.7) \quad \frac{\varepsilon(E - gG_0 - 2mc^2)}{\hbar c} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = (\vec{\sigma} \cdot \vec{P})^2 \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Ignoring the space component $\vec{G} = (G_1, G_2, G_3)$, we deduce from (12.7) that

$$(12.8) \quad -\nabla^2 \Phi + \frac{g}{\hbar c} G_0(x) \Phi = \lambda \Phi,$$

where $\Phi = \psi_1$ and $\lambda = \varepsilon(E - 2mc^2)/(\hbar c)$ represent the energy level of a particle.

We remark here that equation (12.8) can be equivalently derived from the classical Schrödinger equation. For mesons, the corresponding energy equations can be derived from the Klein-Gordon equations.

We now the energy level theory for all elementary particles.

First if all, when we consider the leptons and quarks, we take (12.8) in the following form:

$$(12.9) \quad -\nabla^2 \Phi^w + \frac{g_w}{\hbar c} W(x) \Phi^w = \lambda^w \Phi^w,$$

where $W = \alpha_a^w W_0^a$. If we consider hadrons, we use

$$(12.10) \quad -\nabla^2 \Phi^H + \frac{g_s}{\hbar c} S(x) \Phi^H = \lambda^H \Phi^H,$$

where $S = \alpha_k^s S_0^k$.

Assume that

$$(12.11) \quad \Phi^w, \Phi^H = 0 \quad \text{for } r \geq R, \quad R \text{ is the cosmos radius.}$$

Mathematically it is clear that there are finite number of negative eigenvalues of (12.9) and (12.10) with (12.11), respectively:

$$(12.12) \quad -\infty < \lambda_1^w \leq \lambda_2^w \leq \dots \leq \lambda_K^w < 0,$$

$$(12.13) \quad -\infty < \lambda_1^H \leq \lambda_2^H \leq \dots \leq \lambda_N^H < 0,$$

such that

$$(12.14) \quad \lambda_k^w - \lambda_{k+1}^w \rightarrow 0, \quad \lambda_j^H - \lambda_{j+1}^H \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Let Φ_k^w and Φ_j^H be the corresponding eigenstates of (12.9) and (12.10) respectively. Then we obtain the following assertions:

- Each lepton or quark is represented by an eigenstate Φ_k^w of (12.9) with λ_k^w being its binding energy for some $1 \leq k \leq K$.
- Each hadron is represented by an eigenstate Φ_j^H of (12.10) with λ_j^H being its binding energy for some $1 \leq j \leq N$.
- The eigenstate Φ_1^w of (12.9) with the lowest energy level λ_1^w represents the electron.
- The eigenstate Φ_1^H of (12.9) with the lowest energy level λ_1^H represents the proton.
- A matter particle is regarded as an energy parcel corresponding to an level λ_k^w or λ_j^H , with $|\Phi_k^w|^2$ or $|\Phi_j^H|^2$ being its energy density.

12.2. Particle decays. Based on the energy level theory established above, decay and colliding reactions can be considered as transitions of energy levels. For example, the β -decay

$$n \rightarrow p + e + \bar{\nu}_e$$

is a transition of a neutron in higher energy level to a proton in lower energy level accompanied by the emission of an electron and anti-neutrino, which take away the energy.

Formula (11.22)-(11.24) and (11.29) provide a direct explanation for particle decays. For example, the process that a baryon decays into two hadrons can be regarded as two sub-processes as shown in Figures 12.1 and 12.2, where black dots represent quarks.

Figure 12.1 shows that when externally excited, a pair of quarks in a baryon split with each into two quarks, and the resulting five quarks will immediately form a new baryon and a meson. Figure 12.2(b) illustrates that the two new hadrons are formed causing the decay.

By applying the strong interaction force, the decay process can be interpreted as follows:

- (1). QUARK CONFINEMENT. By (11.22), the quark binding energy is about

$$(12.15) \quad E_q \simeq \frac{g_s^2 B}{r_0^2 \rho_0} \varphi(r), \quad r_0 = k_0^{-1} \simeq 10^{-16} \text{ cm},$$

where $\varphi(r_0) \simeq r_0/2$. The estimated quark radius $\rho_0 \simeq 10^{-21}$ is very small. In addition, by the Yukawa potential, we know that

$$g_s^2 = 10e^2 \simeq 2 \times 10^{-11} \text{ MeV} \cdot \text{cm},$$

and the constant B is estimated in [8] as

$$B \simeq 10^{-2} \text{ cm}.$$

Hence it follows from (12.15) that

$$(12.16) \quad E_q \simeq 10^{21} \text{ GeV},$$

which is beyond the Planck level. Consequently an energy level beyond the Planck energy 10^{19} GeV is required to break free a quark in a baryon.



FIGURE 12.1. Externally excited quarks split in pairs, forming new hadrons.

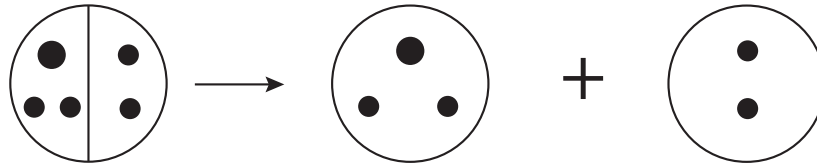


FIGURE 12.2. Two hadrons being push apart after splitting.

(2) When the quarks in a hadron is split forming two or more hadrons, the strong interaction between the two newly formed hadrons follow immediately the strong interaction potential (11.23) for nucleons/hadrons. As these two new hadrons are too close, the strong nuclear force with potential (11.23) is repelling, causing decay.

(3) Quark splitting appears to occur in pairs, i.e. evenness or oddness of the total number of quarks is invariant in a decay process.

13. STABILITY OF MATTER

Based on the theory presented in the previous sections, the structure and stability of matter can be understood in four different scales from the largest cosmos to the smallest elementary particles as follows:

Stars, galaxies and cosmos. Gravity plays the most important role for the structure and its formation of large scale stars, galaxies and cosmos. It is the new gravitational force formula (11.14) established in [10] that shows that gravity is both attracting (Newton and dark matter) for the scale smaller than 10 million light years, and repelling (dark energy) for scale greater than 10 million light years. The largest scale repelling of gravity avoids an eventual collapsing of all galaxies. The attraction of gravity in a relatively smaller scale enables the formation of stars and galaxies.

The dual field φ in the gravitational field equations (11.10) causes the repelling of gravity in the largest scale. It is shown in [10] that the dual φ vanishes if the matter in the universe is uniformly distributed. In summary, it is the interaction between the gravitational field $\{g_{\mu\nu}\}$ and the dual field φ that maintains the large scale structure of the universe.

Atomic and molecular level. Atoms and molecules are held together by Coulomb attracting force. The reasons why atoms do not collapse are mainly due to 1) the energy levels of electrons preventing electrons from collapsing to the nuclear, and 2) the Pauli principle for the stability of bulk matter; see among others [7].

Nucleons/hadron level. By (11.22), the strong interacting force between two quarks is repelling as the distance between them is small avoiding the collapsing quarks together, and is attracting holding them together and forming a hadron as the distance increases. In the hadron level, by (11.23), the strong acting also are repelling when two hadrons are close (less than 10^{-13}cm), again avoiding the collapsing of hadrons. When the distance between two hadrons increases (about $10^{-13} - 10^{-12}\text{cm}$), the strong attracting force takes place binding hadrons together forming a nuclear. Then by (11.24), when the distance between two molecules or atoms is less than 10^{-7}cm , the strong repelling force induces the van der Waals repelling force.

Lepton and quark level. In this level, the acting force is the weak force (11.29). Again the short distance repelling avoids collapsing, and followed by attracting weak force forming a lepton or a quark.

In summary, all four forces display both attracting and repelling hold matter/particle together and avoiding collapsing at the same time. This is the essence of the stability of matter in the universe from the smallest elementary particles to largest galaxies in the universe. Also, the energy levels for leptons and hadrons classifies all leptons and hadrons with electron and proton being the smallest energy level elementary particles.

14. CONCLUSIONS

The main objective of this article is to drive a unified field model coupling four interactions, based on the principle of interaction dynamics (PID) and the principle

of representation invariance (PID). Intuitively, PID takes the variation of the action functional under energy-momentum conservation constraint. PRI requires that physical laws be independent of representations of the gauge groups. One important outcome of this unified field model is a natural duality between the interacting fields (g, A, W^a, S^k) , corresponding to graviton, photon, intermediate vector bosons W^\pm and Z and gluons, and the adjoint bosonic fields $(\Phi_\mu, \phi^E, \phi_w^a, \phi_s^k)$. This duality predicts two Higgs particles of similar mass with one due to weak interaction and the other due to strong interaction. PID and PRI can be applied directly to individual interactions, leading to 1) modified Einstein equations, giving rise to a unified theory for dark matter and dark energy, 2) three levels of strong interaction potentials for quark, nucleon/hadron, and atom respectively, and 3) a weak interaction potential. These potential/force formulas offer a clear mechanism for both quark confinement and asymptotic freedom—a longstanding problem in particle physics. Also, we intend to offer our view on such questions as why our universe is as it is, by introducing energy levels for leptons and quarks as well as for hadrons, and by exploring essential characteristics of the potential/force formulas.

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